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PROCEEDINGS

OF THE

EDINBURGH MATHEMATICAL SOCIETY.

VOLUME V.

SESSION 1886–87.

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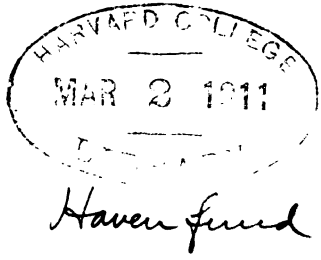

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PROCEEDINGS

OF THE

EDINBURGH MATHEMATICAL SOCIETY.

FIFTH SESSION, 1886-87.

First Meeting, November 13th, 1886.

Dr Ferguson, F.R.S.E., President, in the Chair.

Dr Ferguson, as retiring President, delivered an address, for which and for his kindness in granting the Society free accommodation in the Edinburgh Institution, he was accorded a hearty vote of thanks.

For this Session the following Office-Bearers were elected:—

President—Mr George Thom, M.A.

Vice-President—Mr W. J. Macdonald, M.A., F.R.S.E.

Secretary—Mr A. Y. Fraser, M.A., F.R.S.E.

Treasurer—Mr John Alison, M.A.

Committee—

Messrs R. E. Allardice, M.A.; R. M. Ferguson, Ph.D., F.R.S.E.; George A. Gibson, M.A.; William Harvey, B.A., LL.B.; J. S. Mackay, M.A., F.R.S.E.; Thomas Muir, M.A., LL.D., F.R.S.E.

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Solutions of Euclid's Problems, with a rule and one fixed aperture of the compasses, by the Italian geometers of the sixteenth century.

By J. S. Mackay.

Haec ac similia ad ostentationem ingenii, utilitatem vero pene nullam, inventa sunt.

Hieronymus Cardanus, *De Subtilitate*, Liber XV.

Chasles in his *Aperçu Historique sur l'origine et le développement des Méthodes en Géométrie* (seconde édition, 1875, pp. 214-215) makes the following statement:

"Essays of the same kind as the geometry of the rule and that of the compasses, and which hold, so to speak, the mean between the two, had long previously engaged the attention of famous mathematicians. Cardan first of all in his book *De Subtilitate* had resolved several of Euclid's problems by the straight line and a single aperture of the compasses, as if one had in practice only a rule and invariable compasses. Tartalea was not long in following his rival on this field, and extended this mode of treatment to some new problems. (*General trattato di numeri et misure; 5ta parte, libro terzo;* in-fol. Venise, 1560). Finally, a learned Piedmontese geometer, J.-B. de Benedictis, made it the object of a treatise entitled: *Resolutio omnium Euclidis problematum, aliorumque ad hoc necessario inventorum, una tantummodo circini data apertura;* in-4°. Venise, 1553."

As very little is known of these essays, and as the treatises in which they occur are far from being readily accessible, I have thought it worth while to make an abstract of them. This abstract, if its interest should prove to be historical rather than scientific, may afford some amusement to those who are fond of geometrical curiosities. I have confined myself to the solutions of the problems in Euclid's first six books; those of the problems in the remaining books are not given at all by Cardano, are very briefly dismissed by Benedetti, and in any case are now not worth resuscitation. With reference to the form in which the solutions are presented, it ought perhaps to be said that the sequence of the problems has in all the three cases been rigidly adhered to. The lettering of the figures has frequently been changed, most of the demonstrations have been omitted as unnecessary, such as have been inserted are not always those given by the authors, but the constructions have not been tampered with. When

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the constructions are the same or substantially the same as Euclid's, they have not been reproduced. Before, however, proceeding to the solutions, a word or two must be said in rectification of the statement of Chasles.

In the early part of 1547 Lodovico Ferraro, who was then a public lecturer on mathematics in Milan, and had been a dependant of the celebrated physician and algebraist Hieronimo Cardano (Tartaglia is never tired of calling him "suo creato"), sent an impertinent challenge to Tartaglia, and a brisk war of pamphlets ensued between the two. Thirty-one questions were proposed by each to be answered by the other within a given time, and it was with Tartaglia's questions that the problem arose which forms the subject of the present paper. The questions were first published in the *Seconda Riposta data da Nicolo Tartalea Brisciano a Messer Lodovico Ferraro* (Venice, 21 April, 1547). They are published again, along with their solutions and with the solutions of twenty-two of the thirty-one questions set by Ferraro, in Tartaglia's *General Trattato di Numeri et Misure* (Venice, 1560), Fifth Part, Third Book. Here Tartaglia tells us how the research originated. "I set myself one day (in order not to be unemployed) to try if it was possible to resolve, with any aperture of the compasses proposed by an adversary, the 26th* proposition of the sixth book of Euclid, namely, that where it is proposed to describe a superficies similar to a given rectilineal superficies and equal to another, a thing which not only I soon assured myself was possible, but even found it to be possible to resolve (with such a condition) all his other geometrical problems to be worked on a plane, excepting those where it falls to describe a terminated circle (as is the case in the 4th, 5th, 8th, 9th, 13th, and 14th propositions of his fourth book, and likewise in the 25th and 33rd of the third)."

Ferraro's answers to Tartaglia's questions were first printed in a pamphlet, the heading of which begins, *Quinto Cartello di Lodovico Ferraro contra Messer Nicolo Tartaglia*, and the last part of which is entitled *Risolutione fatto per Lodovico Ferraro à i trentaun' quesiti mandatagli da risolvere per Messer Nicolo Tartaglia* (Milan, October, 1547).

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* This is the 25th proposition of Euclid's sixth book. Tartaglia in his translation of Euclid's *Elements* into Italian (Venice, 1543) now and then changes the order of Euclid's propositions.

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Cardano, in his work *De Subtilitate Libri XXI.* (Norimbergae, 1550) entitles the fifteenth book *De incerti generis aut inutilibus subtilitatibus.* He there, without naming Tartaglia, mentions that both Ferraro and himself found out in a few days how all that Euclid proves, (with the exception of the inscription and circumscription of circles), when the aperture of the compasses may be varied, could be shown by them when the aperture was any invariable one proposed by an adversary. He says that Ferraro printed the demonstration of all this, but as it was for a controversial purpose, he did not think it would survive, and in order that so rare an example of subtilty might not perish, he gives an account of it.

Last of all, Giovanni Battista Benedetti (whose name is Latinised into Joannes Baptista de Benedictis) took the matter up, and published the treatise mentioned by Chasles. In his verbose preface or dedication, Benedetti says nothing about the origin of the problem, or about the published solutions of Ferraro and Cardano.

Tartaglia's Solutions.

1. *On a given straight line to describe an equilateral triangle.*

Euclid I. 1. See figures 1, 2.

Let AB be the given straight line.

Make AC and BD each equal to the given aperture. On AC describe the equilateral triangle ACE, and on BD the equilateral triangle BDF. Let AE and BF intersect at G.

GAB is the equilateral triangle required.

2. *Through a given point to draw a straight line parallel to a given straight line.*

Euclid I. 31. See figures 3, 4.

Let A be the given point, BC the given straight line.

With A as centre cut BC at D. Join AD and produce it, making DE equal to AD. With E as centre cut BC at F. Join EF and produce it, making FG equal to EF. Join AG.

AG is the parallel required.

If the circle with A as centre and the given aperture as radius do not cut BC, choose a point H near enough to BC, and through it draw the parallel HK, as before. Then through A draw AG parallel to HK.

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3. *To bisect a given angle.*

Euclid I. 9. See figure 5.

Let ACB be the given angle.

With C as centre cut CA and CB at D and E. With D and E as centres describe circles intersecting in F. Join CF.

CF is the bisector required.

4. *To bisect a given straight line.*

Euclid I. 10. See figure 6.

Let AB be the given straight line.

With A and B as centres describe circles cutting AB at C and D. If these circles do not intersect, with C and D as centres describe circles; and so on. Finally, let E and F be the points of intersection. Join EF cutting AB at G.

G is the point of bisection.

5. *To draw a perpendicular to a given straight line from a given point in it.*

Euclid I. 11. See figure 7.

Let AB be the given straight line, C the given point.

With C as centre cut CA and CB at D and E. Bisect CD and CE at G and H. With G and H as centres describe circles intersecting in F. Join CF.

CF is the perpendicular required.

6. *To draw a perpendicular to a given straight line from a given point outside it.*

Euclid I. 12. See figure 8.

Let AB be the given straight line, C the given point.

With C as centre cut AB at D and E. With D and E as centres describe circles intersecting in F. Join CF.

CF is the perpendicular required.

If the circle with C as centre and the given aperture as radius do not cut AB, draw a straight line near enough to C and parallel to AB. To that straight line draw a perpendicular from C, and produce the perpendicular to meet AB.

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7. *From a given point to draw a straight line equal to a given straight line.*

Euclid I. 2. See figures 9, 10.

(*a*) Let A be the given point, BC the given straight line. (Fig. 9)

Join AB, and through C draw CD parallel to AB. Through A draw AE parallel to BC meeting CD at E.

AE is the straight line required.

(*b*) Let A be the given point, BC the given straight line. (Fig. 10)

Join AB, and on it describe the equilateral triangle DAB. Produce DA and DB to E and F. With B as centre cut BC and BF at G and H. Join GH, and through C draw CK parallel to GH and meeting BF at K. Through K draw KL parallel to BA and meeting AE at L.

AL is the straight line required.

8. *From the greater of two given straight lines to cut off a part equal to the less.*

Euclid I. 3. See figures 11, 12, 13, 14, 15, 16.

(*a*) Let the given straight lines AB and CD, of which CD is the greater, be parallel. (Fig. 11)

Join AC, and through B draw BE parallel to AC, and meeting CD at E.

CE is the part required.

(*b*) Let the given straight lines AB and AD, of which AD is the greater, be joined at A. (Fig. 12)

With A as centre cut AD and AB at E and F. Join EF, and through B draw BG parallel to FE, and meeting AD at G.

AG is the part required.

(*c*) Let the given straight lines AB and AD, of which AD is the greater, be in the same straight line. (Fig. 13)

From A draw AE perpendicular to AB. From AE cut off AF equal to AB; and from AD cut off AG equal to AF.

AG is the part required.

(*d*) Let the given straight lines AB and CD, of which CD is the greater, be neither parallel, joined at a point, nor in the same straight line. (Fig. 14)

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Through A draw AE parallel to CD. From AE cut off AF equal to AB; and from CD cut off CG equal to AF.

CG is the part required.

(*e*) Let the given straight lines AB and CD, of which CD is the greater, be in the same straight line but not contiguous. (Fig. 15)

From B and C draw BE and CF respectively perpendicular to AB and CD. Cut off BE equal to BA; through E draw EF parallel to AD and meeting CF at F. From CD cut off CG equal to CF.

CG is the part required.

(*f*) Let the given straight lines AB and AC, of which AC is the greater, be joined at A; and let it be required to cut off a part equal to AB from the end C. (Fig. 16)

Join BC, and on it describe the equilateral triangle DBC. Produce DB and DC to E and F. Cut off BE equal to BA; and through E draw EF parallel to BC and meeting DF at F. Cut off CG equal to CF.

CG is the part required.

9. *At a given point in a given straight line to make an angle equal to a given angle.*

Euclid I. 23. See figure 17.

Let A be the given point, AB the given straight line, and CDE the given angle.

With D as centre cut DE at F; with F as centre cut DC at G, and join FG. From AB cut off AH equal to DG; with A and H as centres describe circles intersecting in K. Join KA, KH.

HAK is the angle required.

When angle CDE is obtuse, make, as before, angle HAK equal to its supplement; and find the supplement of angle HAK.

10–14. Euclid I. 42, 44, 45, 46; II. 11.

15. *Given two squares, to apply to one of them a gnomon equal to the other.*

This is not one of Euclid's problems, but is given in Tartaglia's edition of *The Elements* as II. 14. See figure 18.

Let ABCD, EFGH be the two squares.

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Produce AB, and cut off BK equal to EF. Join CK, and from AK cut off AL equal to CK. On AL describe the square ALMN. BCDNML is the gnomon required.

16. Euclid VI. 10.

17. *To find a mean proportional between two given straight lines.*

Euclid VI. 13. See figure 19.

Let AB and BC be the two given straight lines.

Draw any straight line AD making an angle with AC, and cut off AE equal to the given aperture of the compasses. With E as centre describe a circle cutting AD again at F. Join CF, and through B draw BG parallel to CF and meeting AF at G. From G draw GH perpendicular to AF and meeting the circumference at H. Cut off GK equal to GH, and through K draw KL parallel to FC and meeting AC produced at L.

BL is the mean proportional required.

The following demonstration may be given:

On account of the parallels BG, LK, CF,

$$AB : BL : BC = AG : GK : GF,$$
$$= AG : GH : GF.$$

Now GH is a mean proportional between AG and GF;

therefore BL is a mean proportional between AB and BC.

18. *To describe a square equal to a given triangle.*

This is a particular case of Euclid II. 14. It is given as II. 15 in Tartaglia's edition of *The Elements.*

Construct a rectangle equal to the given triangle, and find a mean proportional between its sides. The square on this mean proportional is the square required.

19. *To describe a square equal to a given rectilineal figure.*

Euclid II. 14. It is given by Tartaglia as the last proposition of the second book.

(*a*) The method of solution is the same as that of the preceding problem.

(*b*) Resolve the rectilineal figure into triangles. Find the side of a square equal to the first triangle. At one end of it raise a perpen-

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dicular equal to the side of a square which is equal to the second triangle; and draw the hypotenuse of this right-angled triangle. At one end of the hypotenuse raise a perpendicular equal to the side of a square which is equal to the third triangle; and draw the hypotenuse of this right-angled triangle. Continue this process till all the triangles which make up the rectilineal figure are exhausted. The last hypotenuse is a side of the required square.

20. Euclid III. 1.

21. *From a given point to draw a tangent to a given circle.*

Euclid III. 17. See figure 20.

Let ABG be the given circle, and D the given point.

Find C the centre of the given circle, and draw the straight line DACB. Produce BD to E, and make DE equal to DA.

Divide BA at F, so that BF is to FA as BD to DE; and from F draw FG perpendicular to AB and meeting the circumference at G. Join DG.

DG is the tangent required.

The following demonstration may be given:

Because $BF : FA = BD : DE,$

$= BD : DA;$

therefore BA is divided harmonically at F and D.

Hence FG is the polar of D, and DG a tangent to the circle.

Tartaglia gives no proof of his construction, but refers to Apollonius's *Conics*, Book I., Prop. 34, and adds that in the same manner a tangent may be drawn to the three other conic sections.* He mentions that this problem was the first of the thirty-one questions proposed by him to Cardano and Ferraro in their public dispute.

22–26. Euclid III. 30, 34; VI. 11, 12, 9.

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* Apollonius proves that this construction will give a tangent to the hyperbola, the ellipse, and the circle; but Prop. 35 shows that he knew how to modify it to obtain a tangent to the parabola.

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27. *In a given circle to place a chord equal to a given straight line which is not greater than the diameter of the circle.*

Euclid IV. 1. See figure 21.

Let CDF be the given circle, and AB the given straight line.

Draw any diameter CD, and from it cut off CE equal to the third proportional to CD and AB. From E draw EF perpendicular to CD and meeting the circumference at F. Join CF.

CF is the chord required.

28–38. Euclid IV. 2, 3, 6, 7, 10, 11, 12, 15, 16; VI. 18, 25.

The last of these problems was the second of Tartaglia's questions.

39. *To find a square equal to the difference of two given squares.*

This is not one of Euclid's problems, but is inserted as subsidiary to the problem which follows. See figure 22.

Let ABCD, EFGH be the given squares, the latter being the greater.

Take a straight line LM equal to twice the given aperture, and on it describe a semicircle. Find a fourth proportional to EF, AB, and LM; in the semicircle place a chord LK equal to this fourth proportional; and join KM. Lastly, find PQ a fourth proportional to LM, KM, and EF.

PQ is a side of the square required.

40. *To make a triangle the sides of which shall be equal to three given straight lines, any two of which are greater than the third.*

Euclid I. 22. See figure 23.

Let AB, CD, EF be the three given straight lines.

Take a straight line HK equal to CD. Find a square equal to the sum of the squares on AB and CD; then find a second square equal to the difference between this square and the square on EF. On HK describe a rectangle equal to the half of the second square; and cut off HL equal to the breadth of the rectangle. Find a third square equal to the difference between the squares on AB and HL; from L draw LG perpendicular to HK and equal to a side of this third square; and join GH, GK.

GHK is the triangle required.

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Tartaglia adds that to ensure the perpendicular always falling inside the triangle, the longest side should be chosen for base.

The demonstration follows from Euclid II. 12 or 13.

41. Euclid VI. 28.

This was the third of Tartaglia's questions.

42. *To describe a parallelogram which shall be similar to a given parallelogram and equal to another parallelogram and a triangle.*

This is not one of Euclid's problems, but is inserted as subsidiary to the problem which follows.

43. Euclid VI. 29.

This was the fourth of Tartaglia's questions.

44. Euclid VI. 30.

FERRARO'S AND CARDANO'S SOLUTIONS.

1. Euclid I. 9. Tartaglia's solution.

2. Euclid I. 10. Tartaglia's solution.

3. Euclid I. 11. Tartaglia's solution.

4. *From the greater of two unequal straight lines drawn from the same point to cut off a part equal to the less.*

Particular case of Euclid I. 3. See figure 24.

Let AB and AC be the given straight lines, AC being the greater.

Bisect the angle BAC by AD. With B as centre describe a circle cutting AD at E; with E as centre describe a circle cutting AC at F.

AF is the part required.

If the circle with B as centre does not meet AD, bisect the angles BAD, CAD, and repeat the preceding construction.

5. *On a given straight line to describe an isosceles triangle.*

This is not one of Euclid's problems, but is inserted as subsidiary

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to the problem which follows. The construction is effected by the 2nd and 3rd problems.

6. *From a given point to draw a straight line equal to a given straight line.*

Euclid I. 2. See figure 25.

Let A be the given point, BC the given straight line.

Join AB, and on it describe the isosceles triangle ABD. Produce DB and DA; cut off BE equal to BC, and DF equal to DE.

AF is the straight line required.

7. Euclid I. 3.

8. *At a given point in a given straight line to make an angle equal to a given angle.*

Euclid I. 23. Tartaglia's solution.

If the aperture is too small, which will be the case when the given angle is obtuse, bisect the given angle; at the given point make an angle equal to half the given angle; and repeat the construction.

9. *On a given straight line to describe an equilateral triangle.*

Euclid I. 1.

Let AB be the given straight line.

With the given aperture describe an equilateral triangle; and at A and B make angles equal to two of the angles of this triangle.

10. *To draw a perpendicular to a given straight line from a given point outside it.*

Euclid I. 12.

Let AB be the given straight line, C the given point.

Through C draw CD parallel to AB; and from C draw CE perpendicular to CD.

11. *To find a mean proportional between two given straight lines.*

Euclid VI. 13. See figure 26.

Let AC and CB be the given straight lines.

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Place AC and CB in the same straight line. At A draw AF perpendicular to AB and equal to twice the given aperture. Join BF; and through C draw CE parallel to BF. Make CG equal to EF, and CK equal to AE. On KG, which is equal to twice the given aperture, describe the semicircle KLG; Draw CL perpendicular to KG, and meeting the circumference at L. Join GL; and through B draw BM parallel to GL.

CM is the mean proportional required.

The following demonstration may be given:

Because AE : AC = EF : CB,
= CG : CB,
= CL : CM;
therefore AE : CL = AC : CM;
therefore KC : CL = AC : CM.
Again CL : CG = CM : CB;
and KC : CL = CL : CG;
therefore AC : CM = CM : CB.

12. Euclid II. 14. Tartaglia's solution.

13. *To draw a tangent to a circle from a given external point.*

Euclid III. 17. See figure 27.

Let BGC be the given circle, A the external point.

Through O, the centre of the circle BGC, draw the secant ABC; and find DE a mean proportional between AB and AC. Draw EF perpendicular to DE, and equal to the radius OB; and join DF. At O make the angle AOG equal to the angle DFE; and join AG.

AG is the tangent required.

14–15. Euclid III. 25, 34.

16. *On a given straight line to describe a segment of a circle which shall contain an angle equal to a given angle.*

Euclid III. 33. See figure 28.

Let AB be the given straight line, C the given angle.

With the given aperture describe a circle DEF; and from it cut off a segment DFE containing an angle equal to C. Take any point F in the arc of the segment, and join FD, FE. At A and B make angles BAG, ABG respectively equal to the angles EDF, DEF.

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Then G is a point on the arc of the segment described on AB, and as many other points as may be necessary can be found in a similar manner.

17. *In a given circle to place a chord equal to a given straight line which is not greater than the diameter of the circle.*

Euclid IV. 1. See figure 29.

Let O be the centre of the given circle, CD the given straight line.

Draw any radius OA; and find FG a fourth proportional to OA, CD, and the given aperture. On FG describe an isosceles triangle EFG, whose sides EF, EG are each equal to the given aperture. At O make angle AOB equal to angle FEG; and join AB.

AB is the chord required.

18. *To construct a triangle whose sides shall be equal to three given straight lines, any two of which are greater than the third.*

Euclid I. 22. See figure 30.

Let A, B, C be the three given straight lines, and let A be greater than B, and B greater than C.

With the given aperture describe a circle, and through O its centre draw the diameter DG. Take OE such that A : B = DO : OE; and EF such that B : C = OE : EF.

Again take HK such that FE : EG = DE : HK; and produce HK to L, so that KL may be equal to FE. In the circle place the chord MN equal to HL; from MN cut off MP equal to KL; and join OP. Lastly, at the extremities of A make angles respectively equal to angles OMP and MOP.

The following demonstration may be given:

(*a*) To prove that the point F lies outside the circle.

A, B, and C are proportional to DO, OE, and EF. But B and C are greater than A; therefore OE and EF are greater than DO.

(*b*) To prove that DG is greater than HL.

DE : HK = FE : EG, by construction.

Now DE is greater than EG;

and DE is greater than FE, because DE : FE = A + B : C;

and DE is greater than HK, because FE is greater than EG.

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Hence of the four proportionals DE, HK, FE, EG, the greatest is DE, and consequently the least is EG; therefore DE + EG is greater than HK + FE, that is DG is greater than HL.

(*c*) To prove that MO, OP, PM are respectively equal to DO, OE, EF, and therefore proportional to A, B, C.

MO and DO are radii of the same circle; and PM was made equal to KL or EF. It remains therefore to prove OP equal to OE.

Because $DE : HK = FE : EG$;

therefore $DE \cdot EG = HK \cdot FE$;

$= HK \cdot KL$,

$= NP \cdot PM$,

$= RP \cdot PQ$.

Now DG = RQ; therefore DE = RP, and OE = OP.

Hence the triangle MOP is similar to the triangle whose sides are A, B, C.

19. *To describe an isosceles triangle having each of the base angles double of the vertical angle.*

Euclid IV. 10.

Take any straight line AB, and divide it at C so that $AB \cdot BC$ is equal to AC^2. Construct a triangle whose sides shall be equal or proportional to AB, AB, and AC.

Cardano adds the two following problems:

20. *A diameter of a circle is given, and a point in it. To raise from the given point a perpendicular of such a length that it shall just meet the circumference.*

The length of the perpendicular is equal to a mean proportional between the segments of the diameter.

21. *On a given hypotenuse to construct a right-angled triangle having one of its sides equal to a given straight line.*

See figure 31.

Let AB be the hypotenuse, CD the side.

With the given aperture describe the circle EFG, and draw a diameter EF. To AB, CD, EE find a fourth proportional; in the circle EFG place FG equal to this fourth proportional; and join EG.

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At B make angle ABH equal to angle EFG; cut off BH equal to CD; and join AH.

ABH is the triangle required.

Benedetti's Solutions.

1. *To draw a perpendicular to a given straight line from a given point in it.*

Euclid I. 11. See figure 32.

Let AB be the given straight line, C the given point.

With C as centre describe the semicircle DFE. On CD and CE construct the equilateral triangles CDF, CGE; and join FG. On FG construct the equilateral triangle FHG; and join CH.

CH is the perpendicular required.

2. *To produce a given straight line, which is less than the given aperture, so that the part produced may be equal to the given straight line.*

See figure 33.

Let AB be the given straight line.

At B draw BD perpendicular to AB. With A as centre cut BD at E; and with E as centre cut AB produced at F.

BF is the part required.

3. *To do the same thing when the given straight line is greater than the given aperture.*

See figure 34.

Let AB be the given straight line.

From AB cut off consecutively distances equal to the given aperture, till DB the part that remains is less than the given aperture. Produce DB to E so that BE may be equal to DB. From AE produced, beginning at E, cut off as many distances equal to the given aperture as have been cut off from AB.

4. Euclid I. 10. Tartaglia's solution.

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5. *To draw a perpendicular to a given straight line from a given point outside it.*

Euclid I. 12. See figure 35.

Let AB be the given straight line, C the given point.

Join BC, and produce it so that CD may be equal to BC. Join DA, and bisect it at E. Join CE, and from C draw CF perpendicular to CE.

CF is the perpendicular required.

6. *Through a given point to draw a straight line parallel to a given straight line.*

Euclid I. 31.

Let AB be the given straight line, C the given point.

From B draw BD perpendicular to AB; and from C draw CE perpendicular to BD.

CE is the parallel required.

7. *From a given point to draw a straight line equal and parallel to a given straight line.*

Let AB be the given straight line, C the given point.

Join CB. Through C draw CD parallel to AB, and through A draw AD parallel to BC.

CD is the straight line required.

8. *To cut off from the greater of two given straight lines a part equal to the less, or to produce the less till it is equal to the greater.*

Euclid I. 3. See figure 36.

Let AB and CD be the two given straight lines.

From C draw CE equal and parallel to AB. With C as centre describe a circle cutting CE and CD or these lines produced at F and G. Join FG; and through E draw EH parallel to FG, and meeting CD or CD produced at H.

CH is the straight line required.

9. *To bisect a given angle.*

Euclid I. 9. See figure 37.

Let BAC be the given angle.

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With A as centre describe a circle cutting AB and AC at D and E. Join DE; bisect DE at F; and join AF.

AF is the bisector required.

10. Euclid I. 23. Tartaglia's solution.

11. Euclid I. 42.

12. *On a given* [*indefinite*] *straight line to construct a triangle equal and similar to a given triangle.*

See figure 38.

Let AB be the given straight line, CDE the given triangle.

Cut off BF equal to DC; and make angle FBG equal to angle CDE. Cut off BG equal to DE; and join FG.

FBG is the triangle required.

13–14. Euclid I. 44, 46.

15. *Given two squares, to apply to one of them a gnomon equal to the other.*

Tartaglia's solution.

16–17. Euclid II. 11; VI. 10.

18. Euclid VI. 13. Tartaglia's solution.

19. *To describe a square equal to a given triangle.*

Tartaglia's solution.

20. Euclid III. 1.

21. *In a given circle to place a chord equal to one given straight line, which is less than the diameter, and parallel to another given straight line.*

See figure 39.

Let EKF be the given circle, AB the straight line to which the chord is to be equal, CD the straight line to which the chord is to be parallel.

Through O the centre of the given circle draw the diameter EF

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parallel to CD ; cut off OG and OH each equal to half of AB. Draw GK and HL perpendicular to EF ; and join KL.

KL is the chord required.

22. *With three given straight lines, two of which are equal to each other, and are together greater than the third, to construct a triangle.*

See figure 40.

Let A, B, C be the three given straight lines, of which A and B are equal.

Take any straight line DEF, making DE equal to A, and EF equal to C ; from D draw DG equal to the given aperture. Join EG, and through F draw FH parallel to EG. In a circle whose centre is O, and which is described with the given aperture as radius, place the chord KL equal to GH ; and join OK, OL. Make OM, ON equal to A, B ; and join MN.

OMN is the triangle required.

23. *To construct a triangle equal to a given triangle, and such that it may have an angle equal to a given angle, and a side equal to a given straight line.*

The construction is effected by the 13th problem.

24. *From a given point to draw a tangent to a given circle.*

Euclid III. 17. See figure 41.

Let DBE be the given circle, A the given point.

Find O the centre of the circle ; and join AO cutting the circumference at B. Divide BO at C so that BC : CO = AB : BO ; from C draw CD perpendicular to AO and cutting the circumference at D ; and join AD.

AD is the tangent required.

The following demonstration may be given :

Because $BC : CO = AB : BO$,

therefore $BO : CO = AO : BO$;

therefore $AO \cdot CO = BO^2$.

Hence CD is the polar of A, and AD a tangent to the circle.

25–26. Euclid III. 30, 34.

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27. *To make a triangle the sides of which shall be equal to three given straight lines, any two of which are greater than the third.*

Euclid I. 22.

Benedetti discusses (in six pages) five cases of this problem.

(1) When two of the sides are equal.
(2) When the three sides are equal.
(3) When the square on one side is equal to the sum of the squares on the two other sides.
(4) When the square on one side is less than the sum of the squares on the two other sides.
(5) When the square on one side is greater than the sum of the squares on the two other sides.

In the first case the problem becomes the 22nd; in the second case the 36th; in the third case it reduces to that of constructing a right-angled triangle, having given the sides containing the right angle. The solutions of the fourth and fifth cases, depending on Euclid II. 13 and 12, are substantially the same as Tartaglia's. Benedetti concludes with the remark: "Et contra illos omnes excellentissimos Mathematicos priscos modernosque qui dixerunt impossibile esse hoc problema alio modo posse concludi quam ut docet xxii primi Euclidis, ego vero deo dante labente Anno diuinae incarnationis MDLII Die xv Octobris illud inueni."

28–34. Euclid IV. 2, 3, 6, 7, 10, 11, 12.

35. *In a given circle to inscribe a regular hexagon.*

Euclid IV. 15.

Find O the centre of the circle, and draw the diameter AOD. In the circle place the chord BC equal to the radius and parallel to AD. Join BO, CO, and produce them to meet the circumference in E, F. Join AB, CD, DE, EF, FA.

ABCDEF is the hexagon required.

36. Euclid I. 1. Ferraro's solution.

37. *From a given point in the circumference of a given circle to draw a chord which shall be equal to a given straight line.*

Euclid IV. 1. See figure 42.

Let AB be the given straight line. C the given point.

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Find O the centre of the given circle, and draw the diameter COD. In the circle place the chord EF equal to AB, and parallel to CD. Draw the diameter FOG; and through C draw CH parallel to FG.

CH is the chord required.

38–45. Euclid IV. 26; VI. 11, 12, 9, 18, 25, 28, 29, 30.

46. *On a given hypotenuse to construct a right-angled triangle having one of its sides equal to a given straight line.*

Cardano's solution.

47. *Given two straight lines, one of which is less than half of the other, to divide the greater into two segments such that the smaller straight line shall be a mean proportional between them.*

See figure 43.

Let AB and BC be the two straight lines, BC being less than half of AB.

Place BC at right angles to AB, and join C with O the middle point of AB. With O as centre describe the semicircle DGE; from E draw EF perpendicular to AB and meeting CO at F. Through F draw FG parallel to AB and meeting the semicircle at G; from G draw GH perpendicular to AB. Lastly, divide AB at K so that AK : KB = DH : HE.

AK and KB are the required segments.

The following demonstration may be given:

$$\begin{aligned} EF : BC &= EO : BO, \\ &= 2EO : 2BO, \\ &= DE : AB. \end{aligned}$$

Now EF, which is equal to HG, is a mean proportional between DH and HE, the segments of DE;
therefore BC is a mean proportional between the segments of AB, when AB is divided similarly to DE;
that is, AB is a mean proportional between AK and KB.

48. Euclid IV. 5.

49. *To construct a triangle similar to a given triangle, and such that the centre of its circumscribed circle may be at a given point and the radius of it equal to a given straight line.*

See figure 44.

Let ABC be the given triangle, D the given point, and E the given radius.

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Find F the centre of the circle circumscribed about triangle ABC; and join FA, FB, FC. From D draw DK equal to E; on DK construct triangle DKG similar to triangle FCA, and triangle DKH similar to triangle FCB; and join GH.

GHK is the triangle required.

50–54. Euclid IV. 4, 8, 9, 13, 14.

55. *About a given centre to describe a regular figure which shall be similar to a given regular figure and shall have a given circumscribed radius.*

The method of solution is similar to that of 49.

56. Euclid III. 33.

57. *From a given point to draw a straight line equal to a given straight line.*

Euclid I. 2.

Let A be the given point, BC the given straight line.

From A draw an indefinite straight line AD, and from AD cut off AE equal to BC.

58. *To find the centre of a circle, an arc of which is given.*

Euclid III. 25.

Draw any two chords not parallel to each other. The straight lines which bisect these chords perpendicularly will meet at the required centre.

59. *To find the diameter of a circle which shall be equivalent to two given circles.*

Make the diameters of the two given circles the base and perpendicular of a right-angled triangle. The hypotenuse of this triangle will be the diameter of the required circle.

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Second Meeting, December 10th, 1886.

GEORGE THOM, Esq., President, in the Chair.

Composition de Mathématiques Élémentaires proposée au Concours d' Agrégation de 1886.

Solution par M. PAUL AUBERT.

On donne un cercle et deux points P et Q situés sur un diamètre, on joint les points P et Q aux extrémités A et B d'un diamètre du cercle par les droites PA et QB qui se coupent au point M. On fait tourner le diamètre AB et on demande

I. D' étudier les variations du rapport $\frac{MA}{MB}$ *et de construire la figure quand le rapport a une valeur donnée.*

II. D' étudier les variations de l' angle AMB, et de construire la figure quand cet angle a une valeur donnée.

III. A' et B' étant les seconds points d' intersection des droites MA, MB avec la circonférence donnée, trouver le lieu du centre du cercle circonscrit au triangle MA'B'.

I. On a (fig. 45)

$$\frac{MA}{MB} = \frac{\sin ABM}{\sin BAM}.$$

Menons par le point B la parallèle BR à MP, on a

$$\frac{\sin ABM}{\sin BAM} = \frac{\sin OBQ}{\sin OBR} = \frac{OQ \sin BOQ}{BQ} : \frac{OR \sin BOQ}{BR} = \frac{OQ}{OR} \cdot \frac{BR}{BQ}.$$

Appelons a et b les distances des points P et Q au centre de la circonférence donnée dont nous désignerons le rayon par R, et soit λ l' angle BOQ du diamètre mobile AB avec le diamètre fixe PQ. On a

$$\frac{MA}{MB} = \frac{b}{a} \quad \frac{BR}{BQ} = \frac{b}{a} \sqrt{\frac{a^2 + R^2 - 2aR\cos\lambda}{b^2 + R^2 - 2bR\cos\lambda}}.$$

Si les points P et Q, au lieu d' être de part et d' autre du point O, sont du même côté de ce point (fig. 46), il est facile de voir que

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les relations précédentes subsistent, à condition de changer sous le radical a en $-a$. On est donc amené à étudier les variations de l'expression

$$y = \frac{a^2 + R^2 - 2aR\cos\lambda}{b^2 + R^2 - 2bR\cos\lambda}$$

quand λ varie de 0° à 360°. Il suffira de faire varier λ de 0° à 180°, puisque cos λ prend toutes ses valeurs possibles dans cet intervalle, puis de prendre pour l' intervalle de 180° á 360° les valeurs de l' expression déjà trouvées mais dans l' ordre inverse.

Quand cos λ varie de $+1$ à -1 les deux termes de la fraction y restent toujours positifs. Par suite y ne devient jamais nulle ni infinie. Supposons d' abord P et Q de part et d' autre du centre. On peut écrire

$$\begin{aligned} y &= \frac{a}{b}\,\frac{b(a^2+R^2) - 2abR\cos\lambda}{a(b^2+R^2) - 2abR\cos\lambda} \\ &= \frac{a}{b}\left[1 + \frac{b(a^2+R^2) - a(b^2+R^2)}{a(b^2+R^2) - 2abR\cos\lambda}\right] \\ &= \frac{a}{b}\left[1 + \frac{(R^2 - ab)(b-a)}{a(b^2+R^2) - 2abR\cos\lambda}\right] \end{aligned}$$

Quand cos λ varie de $+1$ à -1 le dénominateur de la fraction augmente constamment. Si son numérateur est positif, y diminue ; si ce numérateur est négatif, y augmente continuellement. Le rapport $\frac{MA}{MB}$ varie de $\frac{b}{a}\,\frac{R-a}{R-b}$ à $\frac{b}{a}\,\frac{R+a}{R+b}$ toujours dans le même sens, la première expression étant prise en valeur absolue.

Si les points P et Q sont du même côté du centre, on a

$$\frac{MA}{MB} = \frac{b}{a}\sqrt{\frac{a^2 + R^2 + 2aR\cos\lambda}{b^2 + R^2 - 2bR\cos\lambda}}.$$

Quand cos λ varie de $+1$ à -1, le numérateur de la fraction diminue, son dénominateur augmente. Donc le rapport diminue continuellement. Il varie de

$$\frac{b}{a}\,\frac{a+R}{b-R} \text{ à } \frac{b}{a}\,\frac{a-R}{b+R}.$$

Dans les deux cas le rapport $\frac{MA}{MB}$ varie toujours dans un certain sens quand λ croît de 0° à 180°, et dans le sens contraire quand λ continue à croître de 180° à 360°.

Construction géométrique : La construction du point M dépend

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de la position du diamètre AB, qui est elle même déterminée par la position du point B. Or nous avons vu que

$$\frac{BR}{BQ} = \frac{a}{b}\frac{MA}{MB}.$$

Si le rapport $\frac{MA}{MB}$ est donné, on connaît donc la valeur du rapport $\frac{BR}{BQ}$. Pour obtenir le point B, on déterminera d' abord les deux points I et I' du diamètre QR (fig. 47) dont le rapport des distances aux points Q et R a la valeur $\frac{a}{b}\frac{MA}{MB}$, puis on décrira sur II' comme diamètre une circonférence qui coupera la circonférence donnée en deux points répondant à la question, si la valeur donnée pour le rapport $\frac{MA}{MB}$ est comprise entre les valeurs limites trouvées précédemment. On peut retrouver ces valeurs limites en discutant les conditions de possibilité de la construction géométrique.

II. Étude de l' angle AMB.

On a dans le triangle BQR dont l' angle B est egal à l' angle AMB

$$\overline{QR}^2 = \overline{BQ}^2 + \overline{BR}^2 - 2BQ \cdot BR\cos B,$$

d' où $$\cos B = \frac{\overline{BQ}^2 + \overline{BR}^2 - \overline{QR}^2}{2BQ \cdot BR}.$$

Mais
$$\overline{BQ}^2 = R^2 + b^2 - 2bR\cos\lambda,$$
$$\overline{BR}^2 = R^2 + a^2 - 2aR\cos\lambda,$$
$$\overline{QR}^2 = (a-b)^2.$$

Effectuant ces substitutions, il vient

$$\cos B = \frac{R^2 + ab - (a+b)R\cos\lambda}{\sqrt{(R^2+b^2)(R^2+a^2) - 2(a+b)(ab+R^2)R\cos\lambda + 4abR^2\cos^2\lambda}}.$$

Il est avantageux de se servir de l' expression de tgB. On trouve sans difficulté en partant de la formule précédente

$$tgB = \frac{(a-b)R\sin\lambda}{R^2 + ab - (a+b)R\cos\lambda}.$$

Posons $tg\frac{\lambda}{2} = t$, il vient

$$tgB = \frac{2(a-b)Rt}{(R+a)(R+b)t^2 + (R-a)(R-b)}.$$

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Nous nous servirons de cette expression pour étudier les variations de l' angle B quand λ varie de 0° à 360°, c'est-à-dire, quand t varie de zéro à $+\infty$ et de $-\infty$ à zéro. Il suffit de faire varier t de zéro à $+\infty$, car quand il varie de $-\infty$ à zéro, tg B reprend les mêmes valeurs dans l' ordre inverse mais changées de signe.

Pour cela considérons la fonction $y = \frac{mt}{pt^2 + q}$.

Si on écrit la relation qui lie y à t sous la forme

$$pyt^2 - mt + qy = 0$$

on voit qu' y ne pourra prendre que les valeurs satisfaisant à l' inégalité

$$m^2 - 4pqy^2 > 0.$$

Deux cas à distinguer 1° $pq < 0$.

L' inégalité est toujours verifiée et y peut prendre toutes les valeurs de $-\infty$ à $+\infty$.

2° $pq > 0$.

Posons $pq = k^2$. L' inégalité s' écrit

$$(m - 2ky)(m + 2ky) > 0$$

et, m et k désignant des quantités positives, on doit avoir

$$-\frac{m}{2k} < y < \frac{m}{2k}.$$

Dans ce cas y reste compris entre deux limites correspondant aux valeurs de t fournies par la relation précédente, où l' on donne successivement à y les deux valeurs $-\frac{m}{2k}$ et $+\frac{m}{2k}$, et en prenant chaque fois pour t la demi-somme des racines de l' equation ainsi obtenue

$$t = -\frac{k}{p} \text{ et } t = +\frac{k}{p}.$$

En résumé, si nous ne considérons que les valeurs positives de t, les variations de y correspondantes sont représentées pour le premier cas par la courbe (fig. 48), et pour le second cas par la courbe (fig. 49), si on a $p > 0$, et par la courbe (fig. 50) si on a $p < 0$.

Revenons maintenant à l' étude de tg B et appliquons les résultats que nous venons d' obtenir.

La condition relative au signe de pq conduit à étudier le signe de $(R + a)(R + b)(R - a)(R - b)$. Cette expression ne change pas quand on y remplace a par $-a$. Donc dans tous les cas de figure elle est du signe de $(R - a)(R - b)$.

1°. $(R - a)(R - b) < 0$, ce qui exprime que l' un des points est

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intérieur et l' autre extérieur à la circonférence. Dans ce cas l' angle B varie de 180° à 90°, puis diminue jusqu' à 0°; il est droit pour

$$\operatorname{tg}\frac{\lambda}{2}=\sqrt{\frac{(a-R)(R-b)}{(R+a)(R+b)}} \quad \text{ou} \quad \cos\lambda=\frac{R^2+ab}{(a+b)R},$$

comme l' indiquait la valeur de cos B.

2°. $(R-a)(R-b)>0$. Les deux points sont alors tous deux extérieurs ou tous deux intérieurs.

Si on a $(R+a)(R+b)>0$, ce qui exclut le cas où les points seraient extérieurs et du même côté, l' angle B est toujours aigu, il croît de zéro à une valeur maxima donnée par

$$\operatorname{tg}B=\frac{(a-b)R}{\sqrt{(R^2-a^2)(R^2-b^2)}}$$

et correspondante à $\operatorname{tg}\frac{\lambda}{2}=\sqrt{\frac{(R-a)(R-b)}{(R+a)(R+b)}}$.

Si on a $(R+a)(R+b)<0$, ce qui exprime que les deux points sont extérieurs à la circonférence et du même côté, l' angle B est toujours obtus, il varie de 180° à un minimum pour augmenter ensuite jusqu' à 180°.

En résumé, quand les points sont l' un intérieur l' autre extérieur à la circonférence, l' angle B varie toujours dans le même sens et prend toutes les valeurs de 0° à 180°.

Quand les points sont tous deux intérieurs, ou tous deux extérieurs l' angle B est toujours soit aigu soit obtus; dans le premier cas, il commence par croître jusqu' à un certain maximum pour décroître ensuite jusqu' à zéro; dans le second cas, il décroît de 180° à un minimum qu' il dépasse ensuite pour retourner à la valeur 180°.

Pour construire géométriquement le point M répondant à une valeur donnée de l' angle M, il suffit de déterminer la position du point B. L' angle QBR étant connu, et les points Q et R étant fixes, on décrira sur RQ un segment de cercle capable de l' angle donné. Les points d' intersection de ce segment avec la circonférence donnée donnent les points B cherchés. On voit que si les points sont l'un intérieur et l'autre extérieur, il y aura toujours au-dessus du diamètre QR un point B répondant à la question. Si les points sont tous deux intérieurs ou tous deux extérieurs, il pourra y avoir deux positions, une seule, ou aucune suivant les cas. L' étude des conditions de possibilité de cette construction géométrique du point B conduirait aux résultats établis plus haut par une autre voie.

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III. Les triangles semblables PMQ et RBQ donnent

$$\frac{BR}{PM}=\frac{QR}{QP}=\frac{QB}{QM};$$

on, puisque BR = PA,

$$\frac{PA}{PM}=\frac{QR}{QP}.$$

Le dernier rapport est constant, puisque les points Q et R sont fixes. Le point P étant fixe, il résulte de cette égalité que le point M est sur une circonférence homothétique à la circonférence donnée par rapport au point P, qui est centre d' homothétie directe.

On a aussi $\frac{QB}{QM}=\frac{QR}{QP}$.

Cette égalité montre que le point Q est le centre d' homothétie inverse des deux circonférences.

Ces résultats se rapportent à la figure 45; ils seraient renversés pour le cas de la figure 46. Dans tous les cas P et Q sont les centres d' homothétie directe et inverse de la circonférence donnée O, et de la circonférence lieu du point M, que nous appellerons circonférence O'. Considérons par exemple la figure 51. Les points M et A' sont anti-homologues. La tangente en A' au cercle O et la tangente en M à la circonférence O' font donc avec MA' des angles égaux, et se coupent en I sur l' axe radical des deux cercles. Pour la même raison, les points M et B' étant anti-homologues, la tangente en B' au cercle O rencontre MI sur l' axe radical, c'est-à-dire au point I, et les angles IMB' et IB'M sont égaux. Le point I est donc situé sur les perpendiculaires élevées au milieu de MA' et au milieu de MB'; c' est donc le centre du cercle circonscrit au triangle MA'B'. Il résulte des constructions précédentes que ce point est toujours sur l' axe radical des circonférences O et O'. Donc le lieu demandé est cet axe radical.

The Equilateral and the Equiangular Polygon.

By R. E. Allardice, M.A.

The Equilateral Polygon.

Since an n-gon is determined by $2n-3$ conditions, and $n-1$ conditions are involved in its being equilateral, there are still in the case of an equilateral n-gon $n-2$ conditions to be determined. These $n-2$ conditions cannot all be given in terms of the angles, since an

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infinite number of n-gons may always be described similar to, but not necessarily congruent with, any given equilateral n-gon. Hence only $n-3$ of the angles of an equilateral n-gon may be assigned arbitrarily, and there must therefore be 3 independent relations connecting the angles of any equilateral n-gon. These three conditions may be obtained by projecting the perimeter of the n-gon on any three lines.

Let a be the common length of the sides; A, B, C.....L, the exterior angles of the n-gon.

Projection on one of the sides adjacent to the angle A gives the relation

$$a\cos A + a\cos(A+B) + a\cos(A+B+C) + \ldots\ldots = 0;$$

$$\therefore \quad \cos A + \cos(A+B) + \cos(A+B+C) + \ldots\ldots = 0. \qquad (1).$$

Similarly projection on the other sides gives

$$\cos B + \cos(B+C) + \cos(B+C+D) + \ldots\ldots = 0, \qquad (2),$$

$$\cos C + \cos(C+D) + \cos(C+D+E) + \ldots\ldots = 0, \text{ \&c.}; \qquad (3),$$

and projection on lines perpendicular to these gives

$$\sin A + \sin(A+B) + \sin(A+B+C) + \ldots\ldots = 0, \qquad (4),$$

$$\sin B + \sin(B+C) + \sin(B+C+D) + \ldots\ldots = 0. \qquad (5).$$

There thus arise $2n$ equations; but, as has been seen, only three of these are independent. This may be proved analytically in the following manner.

Assume equations (1), (2), and (4) above, and assume also

$$\sin B + \sin(B+C) + \sin(B+C+D) + \ldots\ldots = p. \qquad (6).$$

Let $\cos A + i\sin A = e^{iA} = \alpha$, $\cos B + i\sin B = e^{iB} = \beta$, &c.;

then $\cos(A+B) + i\sin(A+B) = e^{i(A+B)} = e^{iA}e^{iB} = \alpha\beta$.

Hence equations (1), (2), (4), and (6) are equivalent to

$$\alpha + \alpha\beta + \alpha\beta\gamma + \ldots\ldots + (\alpha\beta\gamma\ldots\ldots\lambda) = 0,$$

$$\beta + \beta\gamma + \beta\gamma\delta + \ldots\ldots\ldots (\beta\gamma\delta\ldots\ldots\alpha) = pi$$

$$\therefore \quad \alpha - (\alpha^2\beta\gamma\ldots\ldots\lambda) = -\alpha pi;$$

but $\alpha = \cos A + i\sin A$ is not equal to 0,

$$\therefore \quad 1 - (\alpha\beta\gamma\ldots\ldots\lambda) = -pi,$$

$$\therefore \quad \cos(A+B+\ldots\ldots L) + i\sin(A+B+\ldots\ldots L) = 1 + pi,$$

$$\therefore \quad \cos(A+B+\ldots\ldots L) = 1, \ \sin(A+B+\ldots\ldots L) = p,$$

$$\therefore \quad A+B+C\ldots\ldots L = 2n\pi, \ p = 0, \text{ and } \alpha\beta\ldots\ldots\lambda = 1.$$

Hence $\beta + \beta\gamma + \beta\gamma\delta + \ldots\ldots(\beta\gamma\delta\ldots\ldots\lambda\alpha) = 0.$

Hence also, since β is not equal to 0,

$$1 + \gamma + \gamma\delta + \ldots\ldots(\gamma\delta\ldots\ldots\lambda\alpha) = 0,$$

that is, $$(\alpha\beta\gamma\ldots\ldots\lambda) + \gamma + \gamma\delta + \ldots\ldots(\gamma\delta\ldots\ldots\lambda\alpha) = 0,$$

$$\therefore \quad \gamma + \gamma\delta + \ldots\ldots(\gamma\delta\ldots\ldots\lambda\alpha\beta) = 0;$$

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and in the same way the truth of all the other relations may be established.

These equations may also be written in the form

$$1+\beta+\beta\gamma+\ldots\ldots(\beta\gamma\ldots\ldots\lambda)=0, \text{ \&c.}$$

In order now to prove that the equations do in general involve three independent conditions, it will be sufficient to show that in the case where n is 3, they are sufficient to determine the actual values of A, B and C.

The equations may be assumed in the form

$$\cos A+\cos(A+B)+\cos(A+B+C)=0,$$
$$\cos B+\cos(B+C)+\cos(B+C+A)=0,$$
$$A+B+C=2n\pi;$$

which lead at once to the equations

$$1+\cos A+\cos C=0,$$
$$1+\cos B+\cos A=0;$$

whence also

$$1+\cos C+\cos B=0.$$

These equations give $\cos A=\cos B=\cos C=-\frac{1}{2}$; the only solution of which lying between 0 and π is $A=B=C=2\pi/3$.

From the equations in the case of the quadrilateral may be deduced the equations $\cos A=\cos C$ and $\cos B=\cos D$. The only equilateral quadrilateral in the ordinary sense is of course the rhombus; but the above equations include the limiting cases of two straight lines which meet at a point taken twice over, and a single straight line taken four times over.

It should be noticed that in general it is not allowable to take the angles in an arbitrary order. This is exemplified in the case of the quadrilateral; but it may be shown that in general two angles cannot change places.

Suppose that an equilateral polygon may be formed with certain angles taken in the order ABCDEF.....and also with the same angles taken in the order EBCDAF.....

Then $\quad \alpha+\alpha\beta+\alpha\beta\gamma+\alpha\beta\gamma\delta+\alpha\beta\gamma\delta\epsilon+\ldots\ldots=0,$

and $\quad \epsilon+\epsilon\beta+\epsilon\beta\gamma+\epsilon\beta\gamma\delta+\epsilon\beta\gamma\delta\alpha+\ldots\ldots=0$

$\therefore \quad (\alpha-\epsilon)(1+\beta+\beta\gamma+\beta\gamma\delta)=0.$

Hence the angles cannot change places unless they are equal, or the intervening angles satisfy the relation $1+\beta+\beta\gamma+\beta\gamma\delta=0$. This condition involves that the sides between the two vertices considered, themselves form a closed polygon; for it implies that the sum of the projections of these sides on each of two straight lines at right angles

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to one another is zero. This equation which is equivalent to two conditions, since it involves the imaginary unit, only contains $(n-1)$ of the angles in the general case, and is therefore not in itself sufficient to determine that the n angles may be the angles of an equilateral polygon.

It may be noted further that three conditions of the form

$$1+\cos A+\cos(A+B)+\ldots\ldots=0$$

are not sufficient; for they may be satisfied by angles whose sum is not a multiple of 2π. For let n equal lines P_1P_2, P_2P_3.....P_nP_{n+1}, be drawn making angles A, B, C.....K with one another (the first $n-1$ of the angles); then the first condition will be satisfied if P_{n+1} lies in the perpendicular to P_1P_2 through the point P_1. Let now another straight line $P_{n+1}P_{n+2}$ be drawn making with P_nP_{n+1} an angle L; then the second condition will be satisfied if the point P_{n+2} lies in the perpendicular to P_2P_3 through the point P_2; and so on for a third, fourth, etc. condition, the next line making with that last considered an angle A, the next again an angle B, and so on. Thus in the case where n is 3, the conditions are satisfied by the angles of a square, that is, by the values $A=B=C=\pi/2$; in the case where n is 4, the four conditions of the form considered are satisfied by the angles of a regular hexagon, that is, by the values, $A=B=C=D=\pi/3$; and, in the general case, the n conditions are satisfied by the angles of a regular polygon of $2(n-1)$ sides, that is, by the values $A=B=C=\ldots\ldots=\pi/(n-1)$.

The three conditions may be given in the form

$$1+\cos A+\cos(A+B)+\ldots\ldots=0$$
$$1+\cos B+\cos(B+C)+\ldots\ldots=0$$
$$A+B+C+\ldots\ldots=2\pi.$$

Assume

$$1+\sin A+\sin(A+B)+\ldots\ldots=p$$
$$1+\sin B+\sin(B+C)+\ldots\ldots=q.$$

Then

$$1+\alpha+\alpha\beta+\ldots\ldots=pi$$
$$1+\beta+\beta\gamma+\ldots\ldots=qi$$
$$\alpha\beta\gamma..\ \ldots=1.$$

$$\therefore\quad pi-qi\alpha=0;$$
$$\therefore\quad pi-qi(\cos A+i\sin A)=0;$$
$$\therefore\quad q\sin A+(p-q\cos A)i=0.$$

$\therefore$ if sin A is not equal to 0, $q=0$ and $p=0$.

If the equations are assumed in the form

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$$1+\cos A+\cos(A+B)+\ldots\ldots=0$$
$$1+\sin A+\sin(A+B)+\ldots\ldots=0$$
$$A+B+C+\ldots\ldots=2\pi,$$

or in the equivalent form

$$1+\alpha+\alpha\beta+\ldots\ldots=0$$
$$\alpha\beta\gamma\ldots\ldots=1,$$

the truth of all the other equations is at once obvious.

From the above it is seen that when the conditions are given in any of the forms discussed here, either two of the three conditions must be derived from the projection of the sides of the polygon on two different straight lines, or one of them must be that the sum of the exterior angles is $2m\pi$, where m is some integer.

If n angles be found which may be made the angles of an equilateral polygon, then these angles may be combined in various ways so as to form the angles of other equilateral polygons. Suppose, for example, that n is 5; then the conditions connecting the angles may be written

$$1+\alpha+\alpha\beta+\alpha\beta\gamma+\alpha\beta\gamma\delta=0$$
$$\alpha\beta\gamma\delta\epsilon=1;$$

which may be written in the form

$$1+(\alpha\beta)+(\alpha\beta)(\gamma\delta)+(\alpha\beta)(\gamma\delta)(\epsilon\alpha)+(\alpha\beta)(\gamma\delta)(\epsilon\alpha)(\beta\gamma)=0$$
$$(\alpha\beta)(\gamma\delta)(\epsilon\alpha)(\beta\gamma)(\delta\epsilon)=1;$$

and these equations show that an equilateral polygon may be made of which the exterior angles are, $A+B$, $C+D$, $E+A$, $B+C$, $D+E$. In the same way, if n is not divisible by 3, the angles may be added together in sets of three to form the angles of a new equilateral polygon. And it may easily be seen that the angles may be combined in various ways. For let a radius vector rotate from the position OX through an angle A into the position OA, then through an angle B into the position OB, and so on. Now if the radius vector be conceived to start from OX and to rotate into the position of any of the lines just considered, OP say, then form OP into another position, OQ say, and so on, the different angles through which it rotates may be made the angles of an equilateral polygon, it being supposed that the radius vector always rotates in the same direction and never stops in the same position twice. This may be proved in the same way as the particular cases given above; but it is seen even more easily as a consequence of the laws of the composition of vectors.

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THE EQUIANGULAR POLYGON.

In the fact that an n-gon is equiangular $(n-1)$ conditions are involved; and since the other $(n-2)$ conditions may all be given in terms of the sides, it follows that there must be two relations connecting the sides of an equiangular polygon.

The exterior angle of a regular n-gon is $2\pi/n$ or $2m\pi/n = A$, say, where A may be assumed not to be a multiple of π; and hence a is not real and cannot vanish, where

$$a = e^{Ai} = \cos A + i\sin A.$$

Let the sides be denoted by $a, b, c \ldots\ldots k$; and let the perimeter be projected on the side a and on a line perpendicular to a.

Then $$a + b\cos A + c\cos 2A + \ldots\ldots + k\cos(n-1)A = 0;$$
$$b\sin A + c\sin 2A + \ldots\ldots + k\sin(n-1)A = 0.$$

$\therefore$ $$a + ba + ca^2 + \ldots\ldots + ka^{n-1} = 0; \qquad (1)$$
$$a^n = 1.$$

Multiply equation (1) by a^{n-1}; then

$$b + ca + da^2 + \ldots\ldots aa^{n-1} = 0;$$

and in the same way all the other similar equations may be deduced.

It may easily be shown that the two equations may be assumed in the form

$$a + b\cos A + c\cos 2A + \ldots\ldots = 0;$$
$$b + c\cos A + d\cos 2A + \ldots\ldots = 0.$$

The two conditions may be expressed in terms of the sides alone as follows:—

If the sides are represented by $a_1, a_2, a_3 \ldots\ldots a_n$, the conditions are

$$a_1 + a_2 a + a_3 a^2 + \ldots\ldots + a_n a^{n-1} = 0$$
$$a_2 + a_3 a + a_4 a^2 + \ldots\ldots + a_1 a^{n-1} = 0$$
$$\ldots\ldots\ldots\ldots$$
$$a_n + a_1 a + a_2 a^2 + \ldots\ldots + a_{n-1} a^{n-1} = 0.$$

Hence

$$a = -\begin{vmatrix} a_1 & a_2 & \ldots\ldots & a_n \\ a_2 & a_4 & \ldots\ldots & a_1 \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ a_{n-1} & a_1 & \ldots\ldots & a_{n-2} \end{vmatrix} \div \begin{vmatrix} a_2 & a_3 & \ldots\ldots & a_n \\ a_3 & a_4 & \ldots\ldots & a_1 \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ a_n & a_1 & \ldots\ldots & a_{n-2} \end{vmatrix}.$$

But a is not real, while $a_1, a_2, \ldots\ldots a_n$ are all real; and therefore both numerator and denominator in the value of a must vanish. It is obvious that all the other first minors of the circulant determinant $(a_1 a_2 \ldots\ldots a_n)$ must also vanish. The conditions may be expressed in the form—

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$$\begin{Vmatrix} a_2 & a_3 & \ldots\ldots & a_n & a_1 \\ a_3 & a_4 & \ldots\ldots & a_1 & a_2 \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ a_n & a_1 & \ldots\ldots & a_{n-2} & a_{n-1} \end{Vmatrix} = 0.$$

Conversely, if the condition

$$a_1 + a_2\alpha + a_3\alpha^2 + \ldots\ldots a_n\alpha^{n-1} = 0$$

be given, where $\alpha^n = 1$, it may be shown that an equiangular polygon can be formed with the sides a_1, a_2……a_n. For if a polygon be formed with these sides and with $(n-1)$ exterior angles, each equal to $2\pi/n$, the above condition shows that the polygon will be a closed one, and the condition $\alpha^n = 1$ that the last exterior angle will also be $2\pi/n$.

In the reasoning of last paragraph α, instead of being $\cos 2\pi/n + i\sin 2\pi/n$, may have any one of the values $\cos 2k\pi/n + i\sin 2k\pi/n$, where k has any integral value from 0 to $(n-1)$. It would thus appear that there are n distinct species of equiangular n-gons, distinguished by the magnitude of the exterior angle; but some of these are degenerate cases and others are not distinct.

Thus if $$\alpha = \cos 0 + i\sin 0$$

then $$a_1 + a_2 + \ldots\ldots + a_n = 0;$$

and since none of the sides are negative each must be zero.

Again if n is even and

$$\alpha = \cos\pi + i\sin\pi,$$

the condition becomes

$$a_1 - a_2 + \ldots\ldots + a_{n-1} - a_n = 0;$$

and the polygon is a flat one, such as ABCDEFGHA (fig. 52). Further the two values

$$\alpha = \cos 2r\pi/n + i\sin 2r\pi/n,$$
$$\alpha = \cos 2(n-r)\pi/n + i\sin 2(n-r)\pi/n,$$

give the same polygon, the angles regarded as the exterior angles in the two cases being the conjugates of one another.

All the other polygons are distinct; and hence it follows that if n is odd there are $\frac{1}{2}(n-1)$, and if n is even $\frac{1}{2}(n-2)$, equiangular n-gons of different species; all of which, of course, with the exception of one, are crossed polygons.

A particular case of the above is that in which the polygons are regular. In this case, however, if n is not prime, some of the polygons consist of those with a smaller number of sides taken several times over. Thus one of the regular octagons consists of a square

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taken twice over. These may be called degenerate cases; but there are always certain regular n-gons which are not degenerate cases; one such being the ordinary regular non-crossed n-gon. The number of non-degenerate regular polygons is easily seen to be half the number of special roots of the equation $x^n - 1 = 0$. Now if $p, q, r \ldots\ldots$ are primes, and if $n = p^\alpha q^\beta r^\gamma \ldots\ldots$, the number of special roots of this equation is

$$n(1 - 1/p)(1 - 1/q)\ldots\ldots$$

Hence the number of regular n-gons is

$$\tfrac{1}{2}n(1 - 1/p)(1 - 1/q)\ldots\ldots$$

where $p, q, r \ldots\ldots$ are the prime factors of n, including n if n be prime.

The number of regular n-gons may also be seen very easily by consideration of a circle divided into n equal parts. Each point may be joined to the next, or to the next but one, or to the next but two, and so on; and a regular n-gon will be formed in each case provided every point of division is included.

If an equiangular n-gon can be formed with the sides, a_1, $a_2 \ldots\ldots a_n$ taken in a definite order, then an equiangular polygon of any other species may be formed by taking the same sides in a different order, namely in the order, 1st, $(r+1)^{\text{th}}$, $(2r+1)^{\text{th}}$... provided neither of the polygons be of a species which degenerates when it becomes regular. For if an equiangular polygon of the kind considered can be formed with the sides a_1, $a_2 \ldots\ldots a_n$, taken in that order, then

$$a_1 + a_2\alpha + \ldots\ldots + a_n\alpha^{n-1} = 0,$$

where α is a special root of the equation $x^n - 1 = 0$. If β is any other special root of this equation, then

$$\beta = \alpha^r,\ \beta^2 = \alpha^{2r},\ \&c.,$$

$$\therefore \quad a_1 + a_{r+1}\beta + a_{2r+1}\beta^2 + \ldots\ldots = 0;$$

which shows that an equiangular polygon may be formed with the same sides taken in the order, a_1, a_{r+1}, a_{2r+1} ... Figs. 53, 54, 55, represent the three species of equiangular nonagons which can be formed by taking the same lines in different orders. The first is the ordinary non-crossed nonagon the exterior angle of which is $2\pi/9$; the second is formed from the first by taking every second side, and has $4\pi/9$ for its exterior angle; and the third is formed from the first by taking every fourth side, and has $8\pi/9$ for its exterior angle. It is in general impossible to form an equiangular nonagon of the

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remaining (third) species with the same sides, as this species degenerates when it becomes regular, becoming in that case an equilateral triangle taken thrice over. The same thing is indicated by the fact that it is impossible to get all the sides of a nonagon by going round it and taking every third side.

The propositions of last paragraph may also be proved by means of the laws of composition of vectors. For if vectors be drawn through any point parallel to the sides of an equiangular polygon, these vectors will make equal angles with one another, and if they be compounded in the order, 1st, $(r+1)^{\text{th}}$, $(2r+1)^{\text{th}}$..... they will again form an equiangular polygon, provided all the lines are included, when they are compounded in this way, that is, provided n be not a multiple of r.

It may now be shown that conversely an equiangular polygon may be formed with the sides $a_1, a_2 \ldots\ldots a_n$, if the matrix of the circulant $C(a_1 a_2 \ldots\ldots a_n)$ vanishes.

It is sufficient to show that if this matrix vanishes, then

$$a_1 + a_2 \alpha + \ldots\ldots + a_n \alpha^{n-1} = 0, \text{ where } \alpha^n = 1.$$

Now if the above matrix vanishes, then the circulant $C(a_1 a_2 \ldots\ldots a_n)$ or this divided by $(a_1 + a_2 + \ldots\ldots + a_n)$ also vanishes.

But $$C(a_1 a_2 \ldots\ldots a_n) = \Pi(a_1 + a_2 \alpha_r + \ldots\ldots + a_n \alpha_r^{n-1}),$$

where α_r is a root of $x^n - 1 = 0$, and r has every value from 1 to n.

Hence $$a_1 + a_2 \alpha + \ldots\ldots + a_n \alpha^{n-1} = 0,$$

where α is some root of $x^n - 1 = 0$; and therefore an equilateral triangle may be formed with the sides taken in the above order, the exterior angle being one of the values of $2k\pi/n$.

It is obvious that the sides may not be taken in any arbitrary order and that in general only one of the angles $2k\pi/n$ may be taken. As a matter of fact, if the above-mentioned matrix vanishes, at least two of the factors $(a_1 + a_2 \alpha_r + \ldots\ldots + a_n \alpha_r^{n-1})$ must vanish; namely those in which the quantities α_r are conjugate imaginaries. A possible case when n is even, included in the above, is that of the flat polygon mentioned before. This case arises when α is equal to π and the factor which vanishes is then $(a_1 - a_2 + a_3 - \ldots\ldots)$.

The condition that $a_1, a_2 \ldots\ldots a_n$ be the sides of an equiangular polygon may be represented as follows:—

$$\text{Mat.}C(a_1 a_2 \ldots\ldots a_n) = 0.$$

Now, if the sides taken in the order $a_1, a_{r+1}, a_{2r+1} \ldots\ldots$, form an equiangular polygon, which they will do if r is not a factor of n, then

$$\text{Mat.}C(a_1 a_{r+1} a_{2r+1} \ldots\ldots) = 0.$$

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Hence if one of these matrices vanishes so must the other. In fact it may easily be proved by means of the identity

$$C(a_1a_2.....a_n) = \Pi(a_1 + a_2\alpha + a_3\alpha^2.....)$$

that the two circulants are equal; that is the circulant $C(a_1a_2.....a_n)$ is not altered if the letters be written in the order, 1st, $(r+1)^{th}$, $(2r+1)^{th}$ &c., if r be not a factor of n.

It follows from the above that if

$$C(a_1a_2.....a_n) = 0, \text{ and } a_1 + a_2 + ... + a_n \text{ is not equal to } 0,$$

then $$\text{Mat.}C(a_1a_2.....a_n) = 0.$$

For in this case, for some value of α

$$a_1 + a_2\alpha + ... + a_n\alpha^{n-1} = 0;$$

and therefore, as has been seen before, $\text{Mat.}C(a_1a_2.....a_n) = 0$.
It is assumed that if n is even $a_1 - a_2 + a_3 - .....$ is not equal to 0; and that $a_1, a_2.....a_n$ are all real.

Again, if three consecutive sides be increased by x, $-2x\cos A$, x, the polygon still remains equiangular, A being the exterior angle of the polygon. Hence if the first matrix of a circulant vanish and if three consecutive letters p, q, r, be changed to $p+x$, $p-2x\cos A$, $r+x$, where x is arbitrary and A is a certain one of the angles $2k\pi/n$, the circulant will still vanish.

This may also be proved analytically; for let the factor of $C(a_1a_2.....a_n)$ which vanishes be

$$a_1 + a_2\alpha + ...a_n\alpha^{n-1}$$

where $$\alpha = \cos 2p\pi/n + i\sin 2p\pi/n.$$

Then $$a_{r+1}\alpha^r + a_{r+2}\alpha^{r+1} + a_{r+3}\alpha^{r+2}$$

becomes $$a_{r+1}\alpha^r + a_{r+2}\alpha^{r+1} + a_{r+3}\alpha^{r+2}$$
$$+ \alpha^r(x - 2x\alpha\cos 2p\pi/n + x\alpha^2);$$

and $$x - 2x\alpha\cos 2p\pi/n + x\alpha^2$$

$$= x\{1 - 2(\cos 2p\pi/n + i\sin 2p\pi/n)\cos 2p\pi/n + \cos 4p\pi/n + i\sin 4p\pi/n\}$$
$$= x\{(1 - 2\cos^2 2p\pi/n + \cos 4p\pi/n) + i(\sin 4p\pi/n - \sin 4p\pi/n)\}$$
$$= 0.$$

Conversely this transformation will indicate which of the factors of $C(a_1a_2.....a_n)$ vanishes when the circulant itself vanishes.

Particular cases. In the simpler cases the general conditions given above reduce to the following:—

Triangle $$a_1 = a_2 = a_3.$$

Quadrilateral $$a_1 = a_3;\ a_2 = a_4.$$

Pentagon. $$\{4a - (b+c+d+e) + \sqrt{5}(b - c - d + e)\} = 0;$$
$$\{(b-e)^2 + (d-c)^2\} + \sqrt{5}\{(b-e)^2 - (d-c)^2\} = 0.$$

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By means of these equations it may easily be seen that an equiangular pentagon can only have its sides commensurable if it be regular.

Hexagon. $$a_1 + a_2 = a_4 + a_5 ;$$
$$a_2 + a_3 = a_5 + a_6 .$$

Octagon, $$(a_2 - a_4 - a_6 + a_8) + \sqrt{2}(a_1 - a_5) = 0 ;$$
$$(a_2 + a_4 - a_6 - a_8) + \sqrt{2}(a_3 - a_7) = 0.$$

In an equiangular octagon with commensurable sides, opposite sides are equal.

Decagon, $$4(a_1 - a_6) + (a_2 - a_3 + a_4 - a_5 - a_7 + a_8 - a_9 + a_{10}) + \sqrt{5}(a_2 + a_3 - a_4 - a_5 - a_7 - a_8 + a_9 + a_{10}) = 0 ;$$
$$(a_3 + a_5 - a_7 - a_{10})^2 + (a_3 + a_4 - a_8 - a_9)^2 + \sqrt{5}\{(a_3 + a_4 - a_8 - a_9)^2 - (a_2 + a_5 - a_7 - a_{10})^2\} = 0.$$

In an equiangular decagon with commensurable sides,

$$a_1 - a_6 = a_7 - a_2 = a_2 - a_8 = a_9 - a_4 = a_5 - a_{10} ;$$

where a_1 and a_6, a_2 and a_7, &c., are opposite sides.

Dodecagon. $$2(a_1 - a_7) + (a_3 - a_5 - a_9 + a_{11}) + \sqrt{3}(a_3 - a_6 - a_8 + a_{12}) = 0 ;$$
$$2(a_4 - a_{10}) + (a_2 + a_6 - a_8 - a_{12}) + \sqrt{3}(a_8 + a_5 - a_9 - a_{11}) = 0.$$

In an equiangular dodecagon with commensurable sides,

$$a_2 - a_8 = a_{10} - a_4 = a_6 - a_{12} ; \; a_1 - a_7 = a_9 - a_3 = a_5 - a_{11} ;$$

where a_2 and a_8, a_4 and a_{10}, &c., are opposite sides.

Third Meeting, January 14th, 1887.

W. J. MACDONALD, Esq., M.A., Vice-President, in the Chair.

On Certain Inverse Roulette Problems.

By PROFESSOR CHRYSTAL.

The problem of designing cams or centrodes to produce any given motion in one plane is one of some practical importance; and it seems worth while to illustrate by examples some simple methods by which the solution can in certain cases be arrived at. These methods are founded, for the most part, on the use of the so-called Pedal Equation (or p-r-equation), which has great advantages in the present investigation, inasmuch as it depends on the form but not on the position of the curve which it represents.

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Few of the results arrived at are absolutely new. Most of them have been found directly by Clerk Maxwell in a paper published in the Transactions of the Royal Society of Edinburgh, Vol. XVI., 1849.

If we suppose a plane Π to slide upon a fixed plane Π', it is obvious that the motion of Π is determined if the space loci of two points of Π, say P and Q, be given. There will therefore be an infinite number of ways of causing Π to move so that any point P in it may have a given locus. In other words, the problem to generate a given plane curve as a roulette is indeterminate.

I. If, however, there be given a fixed curve C', then it is a determinate problem to find what curve C in Π must roll on C' in order that the point P in Π may trace a given curve R.

II. Again, if there be given a curve C in Π, then it is a determinate problem to find a curve C' such that, if C roll on C', then P shall trace a given curve R.

Owing to the fact that the point P is fixed in Π, and thus affords an origin of reference for the curve C, or body centrode, as it is called, the first of these problems is in general easier than the second. We can at all events in general find, without much difficulty, an equation connecting the radius vector (r) from P and the perpendicular (p) from P on the tangent to C.

In fact, if K (Fig. 56) be the point of contact of the body and space centrodes C and C', then, if P be the corresponding position of the point which traces out the given curve R, we know that PK is normal to R. Moreover, the tangent to C' at K is the tangent to C at K. Hence, if PM be perpendicular to this tangent, we have (P being fixed with reference to C) $PK = r$, $PM = p$.

Now the two curves C' and R are given in the plane Π'; and from their properties we can deduce a relation between PM, and PK, P being a variable point on R, and PK normal to R.

If this relation be $f(PM, PK) = 0$, then the p-r-equation to the curve C, with respect to P as origin, will be

$$f(p, r) = 0 \qquad (1).$$

Since $1/p^2 = 1/r^2 + (dr/r^2 d\theta)^2$, (1) gives us the differential equation

$$f\left\{\left(\frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2\right)^{-\frac{1}{2}}, \; r\right\} = 0 \qquad (2).$$

The obtaining of the relation (1) is a comparatively simple matter in many cases; but as a rule the integration of the equation (2) presents great difficulty.

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Cases where the Space Centrode is a Straight Line.

In such cases, if we take the given straight line as the x-axis, we have merely to find an equation between the normal PG and the ordinate PN to the given curve R; and this relation will furnish at once the p-r-equation to the body centrode, if we put $PN = p$, and $PG = r$.

Generation of a circle, the space centrode being any straight line. (Fig. 57).

Let a be the radius of the circle, b the distance of its centre A from the given straight line C'. We have

$$p = PN = b + a \sin\phi,$$
$$r = PG = a + b \operatorname{cosec}\phi,$$

ϕ being the angle PGD.

Hence
$$(p-b)(r-a) = ab,$$

or
$$\frac{a}{r} + \frac{b}{p} = 1. \qquad (1).$$

From (1) we deduce

$$\frac{a^2}{r^2} - \frac{2a}{r} + 1 = b^2\left(\frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{a\theta}\right)^2\right).$$

Hence
$$d\theta = \pm \frac{b\,dr}{r^2\sqrt{\left(1 - \frac{2a}{r} + \frac{a^2-b^2}{r^2}\right)}} \qquad (2).$$

If $a > b$ we have

$$d\theta = \mp \frac{b}{\sqrt{a^2-b^2}} \frac{d\left(\frac{1}{r} - \frac{a}{a^2-b^2}\right)}{\sqrt{\left\{\left(\frac{1}{r} - \frac{a}{a^2-b^2}\right)^2 - \frac{b^2}{(a^2-b^2)^2}\right\}}},$$

$$\chi = \frac{\sqrt{a^2-b^2}}{b}\theta + \alpha = \log\left\{\frac{1}{r} - \frac{a}{a^2-b^2} + \sqrt{\left(\frac{1}{a^2-b^2} - \frac{2a}{(a^2-b^2)}r + \frac{1}{r^2}\right)}\right\},$$

$$e^{\chi} - \left(\frac{1}{r} - \frac{a}{a^2-b^2}\right) = \sqrt{(\&c.)},$$

$$e^{2\chi} - 2\left(\frac{1}{r} - \frac{a}{a^2-b^2}\right)e^{\chi} + \left(\frac{1}{r} - \frac{a}{a^2-b^2}\right)^2 = \frac{1}{r^2} - \frac{2a}{(a^2-b^2)r} + \frac{1}{a^2-b^2},$$

$$\frac{1}{2}\left(e^{\chi} + \frac{b^2}{(a^2-b^2)^2}e^{-\chi}\right) = \frac{1}{r} - \frac{a}{a^2-b^2}.$$

This we may write

$$\frac{1}{2}e^{\alpha}\left(e^{c\theta/b} + \frac{b^2e^{-2\alpha}}{(a^2-b^2)^2}e^{-c\theta/b}\right) = \frac{1}{r} - \frac{a}{c^2},$$

where $c = \sqrt{a^2-b^2}$.

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Since a depends merely on the choice of the prime radius vector through P, we may so select the latter that $b^2e^{-2a}/(a^2-b^2)^2=1$, we then have

$$\frac{1}{2}\frac{b}{c^2}\left(e^{c\theta/b}+e^{-c\theta/b}\right)=\frac{1}{r}-\frac{a}{c^2},$$

$$r=\frac{c^2}{a+b\cosh(c\theta/b)}.$$

If $a<b$, (2) may be written

$$d\theta=+\frac{b}{\sqrt{b^2-a^2}}\frac{d\left(\frac{1}{r}+\frac{a}{b^2-a^2}\right)}{\sqrt{\left\{\frac{b^2}{(b^2-a^2)^2}-\left(\frac{1}{r}+\frac{a}{b^2-a^2}\right)^2\right\}}}.$$

Hence

$$a+\frac{c\theta}{b}=\cos^{-1}\left(\frac{\frac{c^2}{r}+a}{b}\right);$$

where $c=\sqrt{b^2-a^2}$.

We may choose the prime radius so as to annul a; hence we have

$$b\cos\frac{c\theta}{b}=\frac{c^2}{r}+a,$$

or

$$r=\frac{c^2}{b\cos(c\theta/b)-a}.$$

If $a=b$, (2) becomes

$$d\theta=+\frac{ad\left(\frac{1}{r}\right)}{\sqrt{\left(1-\frac{2a}{r}\right)}},$$

$$\theta+\alpha=\sqrt{\left(1-\frac{2a}{r}\right)};$$

which, by properly choosing the prime radius, we may write

$$r=\frac{2a}{1-\theta^2}.$$

Generation of a straight line, the space centrode being a straight line inclined at an angle α to the given straight line. (Fig. 58).

Here we have

$$\mathrm{PG}\cos\alpha=\mathrm{PN};$$

$$r\cos\alpha=p.$$

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Hence the body centrode is an equiangular spiral whose pole is P, and whose angle is $\frac{1}{2}\pi - a$.

Generation of an ellipse, the space centrode being the major axis. (Fig. 59).

If the co-ordinates of P be (x, y), so that $y^2 = (1-e^2)(a^2-x^2)$,

we have $p^2 = y^2 = (1-e^2)(a^2-x^2)$;

and $r^2 = PG^2 = NG^2 + PN^2$,

$= (1-e^2)^2x^2 + y^2$,

$= (1-e^2)(a^2-e^2x^2)$.

Hence $r^2 - e^2p^2 = (1-e^2)^2a^2$;

that is
$$p^2 = \frac{a^2}{a^2-b^2}r^2 - \frac{b^4}{a^2-b^2} \qquad (1).$$

Now (see Williamson Diff. Calc. p. 346) if a be the radius of the fixed, β the radius of the rolling circle of an epicycloid, and if the centre of the fixed circle be the origin, the p-r-equation is
$$p^2 = \frac{(a+2\beta)^2}{4\beta(a+\beta)}r^2 - \frac{a^2(a+2\beta)^2}{4\beta(a+\beta)} \qquad (2).$$

Comparing this with (A) we see that they will agree provided
$$\frac{(a+2\beta)^2}{4\beta(a+\beta)} = \frac{a^2}{a^2-b^2}, \quad \frac{a^2(a+2\beta)^2}{4\beta(a+\beta)} = \frac{b^4}{a^2-b^2}.$$

This gives
$$a = \frac{b^2}{a}, \quad \beta = \frac{1}{2}\frac{b}{a}(a-b) \text{ or } = -\frac{1}{2}\frac{b}{a}(a+b).$$

Hence any ellipse can be generated by the rolling of an epicycloid, or of a hypocycloid, upon its major axis, the tracing point being the centre of the fixed circle. The radius of the fixed circle is the radius of curvature at the end of the major axis; and the radius of the generating circle is half the difference, or half the sum, of this radius of curvature and of the semi-axis minor.

Cor. 1.—A parabola can be described as a roulette by the rolling of the involute of a circle upon its axis, the tracing point being the centre of the circle, and the radius of the circle being half the latus rectum.

Cor. 2.—The p-r-equation of the body centrode for a hyperbola in case above considered is
$$p^2 = \frac{a^2+b^2}{a^2}r^2 - \frac{b^2}{a^2+b^2}.$$

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Cor. 3.—If the space centrode be the minor axis of the ellipse, the body centrode is one or other of the hypocycloids given by

$$\alpha = a^2/b,\ \beta = \tfrac{1}{2}a(a \pm b)/b.$$

Generation of a parabola, the space centrode being the directrix. (Fig. 60).

We have here, $4a$ denoting the latus rectum of the given parabola,

$$\frac{NG}{p} = \frac{PM}{MH}$$

$$= \frac{\sqrt{4a(p-a)}}{2a}$$

$$NG = p\sqrt{\left(\frac{p}{a} - 1\right)}.$$

Hence

$$r^2 = p^2 + p^2\left(\frac{p}{a} - 1\right);$$

that is,

$$ar^2 = p^3 \qquad (1).$$

This leads to the differential equation

$$d\theta = \pm \frac{a^{\frac{1}{3}}dr}{r\sqrt{\left(r^{\frac{2}{3}} - a^{\frac{2}{3}}\right)}}$$

Hence, by properly choosing the prime radius, we have

$$\theta = 3\sin^{-1}\left(\frac{a}{r}\right)^{\frac{1}{3}},$$

that is

$$r = a\,\mathrm{cosec}^3\frac{\theta}{3} \qquad (2);$$

or, in Cartesian coordinates,

$$27a(x^2 + y^2) = (4a + y)^3 \qquad (2^1).$$

The form of this cubic is given in Fig. 61.

Cor.—If we write R in place of p, and R^2/P in place of r, in equation (1), we obtain the equation to the pedal of the generating cubic. The result is

$$P^2 = a\,R,$$

which represents a parabola identical with the given parabola.

Hence the body centrode for a parabola, when the space centrode is the directrix, is the first negative pedal of the given parabola, with respect to its focus, the tracing point being the focus.

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Generation of a straight line, the space centrode being a circle. (Fig. 62).

Let $CA = d$ be the perpendicular from the centre of the circle on the given line; and let $KCA = 0$.

Then, if a be the radius of the circle, we have

$$r = PK = a\cos\theta - d;$$
$$p = r\cos\theta.$$

Hence $r^2 = ap - dr$,

that is, $ap = r^2 + dr$ (1).

This equation can be integrated; but we confine ourselves here to the case where $d = 0$. We then have

$$ap = r^2 \quad (2).$$

This is the p-r-equation to a circle where diameter is a, the origin being a point on the circumference. Hence the straight line is generated by a circle whose diameter is equal to the radius of the given circle, the tracing point being on the circumference, a very familiar result (Fig. 63). It is to be observed, however, that it is only the part DE of the line that can be thus generated, and that it will be impossible to generate the rest of the line with the given circle as space centrode. Similar limitations will obviously occur in the more general case where d is not zero.

Generation of the cardioid, the space centrode being a circle passing through the cusp, whose centre lies on the axis of the curve, and whose diameter is half the length of the axis. (Fig. 64).

The polar equations to the cardioid and to the circle are

$$r = a(1 + \cos\theta) \text{ and } r = a\cos\theta$$

respectively.

Since for the cardioid $rd\theta/dr = -\cot\frac{1}{2}\theta$, if PK be the normal, we have $OPK = \frac{1}{2}\theta$.

Let OK bisect the angle POB. The $OK = KP$, and we have $2OK\cos\frac{1}{2}\theta = OP = 2a\cos^2\frac{1}{2}\theta$.

Hence $OK = a\cos\frac{1}{2}\theta$; and it follows that K lies on the circle.

If C be the centre of the circle $CKO = COK = \frac{1}{2}\theta$. It follows that CK is parallel to OP. Hence the tangent to the circle at K is perpendicular to OP.

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We have therefore

$$p = \mathrm{PM} = a\cos^2\tfrac{1}{2}\theta\,;$$
$$r = \mathrm{PK} = a\cos\tfrac{1}{2}\theta.$$

Hence $$ap = r^2 \qquad (1).$$

Now (1) is the p-r-equation to a circle of diameter a, the origin being a point on the circumference. Hence the body centrode in the present case is a circle whose diameter is the same as that of the fixed circle, and the tracing point is a point on its circumference.

Cor.—We have, incidentally, a proof that the pedal of a circle with respect to a point on its circumference is the cardioid generated by a pair of circles whose diameters are each equal to the radius of the given circle.

For M is a point on the pedal of the circle OKB; and OM = MP.

Hence the locus of M is a cardioid similar to OPA, the ratio of similarity being 1 : 2.

Generation of an ellipse, the space centrode being the major auxiliary circle. (Fig. 65).

Here PK = r, PM = p. Denote PH, the perpendicular from P on the conjugate diameter by P. Then, with the usual notation, we have

$$\mathrm{KH}^2 - \mathrm{HP}^2 = \mathrm{CK}^2 - \mathrm{CP}^2,$$

that is
$$(r - \mathrm{P})^2 - \mathrm{P}^2 = a^2 - \mathrm{CP}^2$$
$$= \mathrm{CD}^2 - b^2$$
$$= a^2b^2/\mathrm{P}^2 - b^2 \qquad (1).$$

Also, since $$\mathrm{KCH} \approx \mathrm{PKM},$$
$$p/r = (r - \mathrm{P})/a.$$

Hence $$\mathrm{P} = r - ap/r = (r^2 - ap)/r \qquad (2).$$

We have therefore, from (1) and (2)

$$r^2 - 2(r^2 - ap) = a^2b^2r^2/(r^2 - ap) - b^2\,;$$

that is $$2(r^2 - ap)^3 - (r^2 + b^2)(r^2 - ap)^2 + a^2b^2r^2 = 0 \qquad (3).$$

This is the p-r-equation to the body centrode. The derivation of the polar equation would obviously be difficult.

GENERATION OF A GIVEN CURVE AS A ROULETTE WHEN THE BODY CENTRODE IS GIVEN.

Referring back to figure 56, we see that we have now the curve C given in the plane Π, and R given in the plane Π′. P being a point

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fixed with respect to C, we may suppose the p-r-equation of C given with respect to P as origin say

$$f(P, R) = 0 \qquad (1).$$

The problem now is, given a curve (Fig. 66) to determine the locus of a point K on the normal at P which is such that, if PM be perpendicular to the tangent of this locus at K and PM = P, PK = R, then the relation (1) shall be satisfied.

Let the co-ordinates of P and K, with reference to any pair of rectangular axes be (X, Y) and (x, y) respectively; and let the equation to (R) be

$$\phi(X, Y) = 0 \qquad (2).$$

Then, since PK is normal to (R), we have

$$(X - x)/\phi_X = (Y - y)/\phi_Y \qquad (3).$$

Also, if p denote dy/dx, we have

$$P = \{(X - x)\, p - (Y - y)\}/\sqrt{(1 + p^2)} \qquad (4),$$

And, finally,

$$R^2 = (X - x)^2 + (Y - y)^2 \qquad (5).$$

Between these five equations we can eliminate P, R, X, Y. The resultant is a differential equation of the first order connecting x and y, the integration of which will give the equation in Cartesian co-ordinates to the required space centrode.

Generation of a straight line, the body centrode being a straight line.

In this case the equations of the general theory reduce as follows:—

$$P = c \qquad (1),$$

where c is the distance of the tracing point from the rolling straight line.

$$Y = 0 \qquad (2),$$

if we take the generated straight line as axis of X.

$$X - x = 0 \qquad (3).$$

$$P = \{(X - x)p - (Y - y)\}/\sqrt{(1 + p^2)} \qquad (4).$$

$$R^2 = (X - x)^2 + (Y - y)^2 \qquad (5).$$

The result of the elimination is

$$c\sqrt{(1 + p^2)} = y \qquad (6).$$

This gives

$$c\,dy/\sqrt{(y^2 - c^2)} = dx.$$

Hence

$$\log(y + \sqrt{y^2 - c^2}) = x/c + A;$$

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or, if we so choose the origin that $x = o$ when $y = c$,

$$\log \frac{y + \sqrt{y^2 - c^2}}{c} = x/c.$$

This gives

$$y = \frac{c}{2}(e^{x/c} + e^{-x/c}).$$

Hence the space centrode is a catenary, whose directrix is the axis of x.

In point of fact if KM be the tangent at any point K of the catenary (Fig. 67) KP the ordinate, and PM perpendicular to KM, then we have, as is well known, PM = c and KM = arc AK, this verifies the result we have just arrived at.

Generation of a circle, the body centrode being a straight line.

Let the distance of the tracing point from the line be b; and the radius of the circle a; then the equations required to determine the space centrode are as follows:—

$$P = b \tag{1}.$$

$$X^2 + Y^2 = a^2 \tag{2}.$$

$$(X - x)/X = (Y - y)/Y \tag{3}.$$

$$P = \{(X - x)p - (Y - y)\}/\sqrt{(1 + p^2)} \tag{4}.$$

These lead to

$$(xp - y)\{a/\sqrt{(x^2 + y^2)} - 1\} = b\sqrt{(1 + p^2)} \tag{6}.$$

or

$$x^2 d\left(\frac{y}{x}\right)\{a/\sqrt{(x^2 + y^2)} - 1\} = b\sqrt{(dx^2 + dy^2)}.$$

Changing to polar co-ordinates this equation becomes

$$r^2 d\theta\left(\frac{a}{r} - 1\right) = b\sqrt{(dr^2 + r^2 d\theta^2)}$$

which gives

$$d\theta = \frac{\pm b dr}{r^2\sqrt{\left\{1 - \frac{2a}{r} + \frac{a^2 - b^2}{r^2}\right\}}}$$

This is identical with the equation obtained above for the body centrode in the generation of a circle when the space centrode is a straight line.

We have, therefore, a remarkable reciprocity between the two cases, viz.:—If C be the body centrode for generating a circle when the space centrode is a straight line at a distance b from the centre,

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then C will be the space centrode for generating the same circle when the body centrode is a straight line, and the tracing point is at a distance b from A.

Generation of any Curve by Means of Identical Centrodes.

It is here understood that the body and space centrodes are to be congruent curves, and that the point of contact is always to be an identically corresponding point of the two curves.

When the position of a certain point (Q) in the plane of the given curve (R) is assumed, the problem of describing R by means of identical centrodes is determinate. In fact, either centrode is similar to the first negative pedal of R, the ratio of similarity being 1 : 2.*

This appears at once (Fig. 68) if we reflect that in the present case the space centrode C′ is the image of the body centrode C in the common tangent K. For if Q be the image of P in the tangent KM, then since P is a fixed point relative to C, Q will be a fixed point relative to C′. In other words, Q is a fixed point relative to R. Moreover, since $QM = \frac{1}{2}QP$, the locus of M is similar to R, and the locus of M is the pedal of C′, with reference to Q; or, what comes evidently to the same thing, the locus of P is the pedal of a curve similar to C′, the ratio of similarity being 2 : 1.

Hence, when the point Q is selected, the determination of the centrodes is complete.

We may, of course, regard C′ as the envelope of the perpendicular bisector of QP.

In many cases this is the simplest way of treating the problem.

We may, however, use the p-r-equation of the given curve (R).

If this equation be

$$f(p, r) = 0, \qquad (1),$$

the p-r-equation to the locus of M is

$$f(2P, 2R) = 0.$$

Hence, if (p, r) be the p-r-co-ordinates of K, we have

$$P/R = p/r, \text{ and } R = P;$$

that is $P = p^2/r$, and $R = P$.

The p-r-equation of the centrode is therefore

$$f(2p^2/r, 2p) = 0 \qquad (2).$$

---

* This theorem seems to have been first pointed out by Steiner.

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Generation of a straight line by means of identical centrodes.

The equation to R in this case is

$$p = 2a \qquad (1),$$

Where $2a$ is the distance of any chosen fixed point Q from the line.

The equation to either centrode is therefore

$$2p^2/r = 2a\,;$$

that is,

$$p^2 = ar \qquad (2).$$

This represents a parabola, of which Q is the focus.

Hence we conclude that a straight line may be traced by any parabola rolling on an equal parabola, the tracing point being the focus, a well-known result.

Generation of a circle by means of identical centrodes (*Fig* 69).

If c be the radius of the circle, and Q the chosen position of Q at a distance d from the centre O, then the p-r-equation to the circle is

$$p = (r^2 + c^2 - d^2)/2c \qquad (1).$$

Hence the p-r-equation to the centrode is

$$2p^2/r = (4p^2 + c^2 - d^2)/2c.$$

That is

$$\frac{1}{p^2} = \frac{4}{c^2 - d^2}\left(\frac{c}{r} - 1\right) \qquad (2).$$

Comparing this with the focal p-r-equation of an ellipse, viz.—

$$\frac{1}{p^2} = \frac{1}{b^2}\left(\frac{2a}{r} - 1\right),$$

we see that either centrode is an ellipse whose major axis is c, and whose minor axis is $\sqrt{c^2 - d^2}$.

It is easy to see that the major axis of the space centrode must be along QO; and since we have $\sqrt{(a^2 - b^2)} = d/2$, we see that the centre of the ellipse bisects QC.

Generation of an ellipse by identical centrodes.

Take Q at the focus.

$$\frac{1}{p^2} = \frac{1}{b^2}\left(\frac{2a}{r} - 1\right)$$

$$\frac{r^2}{4p^4} = \frac{1}{b^2}\left(\frac{a}{p} - 1\right)$$

This is the polar reciprocal of a limaçon with respect to its double point.

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Generation of a parabola by identical centrodes.

Take Q at the focus.

$$p^2 = ar,$$

the

$$\frac{4p^4}{r^2} = 2ap$$

$$p^3 = ar^3.$$

This is the polar reciprocal of a cardioid with respect to its cusp.

Generation of the lemniscate by identical centrodes.

Take Q at the double point.

$$a^2 p = r^3$$

$$a^2 \frac{2p^2}{r} = 8p^3$$

$$\tfrac{1}{4}a^2 = pr$$

This is an equilateral hyperbola.

Generation of an equilateral hyperbola by identical centrodes.

Take Q at the centre.

$$pr = a^2$$

$$\frac{4p^3}{r} = a^2$$

$$p^3 = \tfrac{1}{3}a^2 r.$$

This is the polar reciprocal of the lemniscate.

Trigonometrical Mnemonics.

By William Renton.

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Fourth Meeting, February 11th, 1887.

GEORGE THOM, Esq., President, in the Chair.

Proof of a Geometrical Theorem.

BY W. J. MACDONALD, M.A.

The middle points of the diagonals of a complete quadrilateral are collinear.

Let ABCDEF be a complete quadrilateral (see fig. 70), and X, Y, Z, the middle points of its diagonals. To prove X, Y, Z, collinear.

Let PQR be the diagonal triangle.

Then BPDQ is a harmonic range.

$$\therefore \quad BX^2 = XP.XQ$$

$$\frac{BP}{BQ} = \frac{BX+XP}{BX+XQ} = \frac{\sqrt{XP.XQ}+XP}{\sqrt{XP.XQ}+XQ} = \frac{\sqrt{XP}(\sqrt{XQ}+\sqrt{XP})}{\sqrt{XQ}(\sqrt{XP}+\sqrt{XQ})} = \frac{\sqrt{XP}}{\sqrt{XQ}}$$

Similarly, $$\frac{AP}{AR} = \frac{\sqrt{YP}}{\sqrt{YR}} \text{ and } \frac{QF}{FR} = \frac{\sqrt{ZQ}}{\sqrt{ZR}}$$

Now $$\frac{PB}{BQ}.\frac{QF}{FR}.\frac{RA}{AP} = -1 \quad (\because \text{ triangle PQR is cut by BF})$$

i.e., $$\frac{BP}{BQ}.\frac{FQ}{FR}.\frac{AR}{AP} = 1$$

i.e., $$\sqrt{\frac{XP}{XQ}.\frac{ZQ}{ZR}.\frac{YR}{YP}} = 1$$

$$\therefore \quad \frac{XP}{XQ}.\frac{ZQ}{ZR}.\frac{YR}{YP} = 1$$

$$\therefore \quad \frac{PX}{XQ}.\frac{QZ}{ZR}.\frac{RY}{YP} = -1$$

$\therefore$ X, Y, Z, are collinear.

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On Vortices.

By C. Chree, M.A.

Helmholtz and most, if not all, subsequent writers on vortex motions have, except in obtaining the fundamental equations, confined themselves to fluid of invariable density.

In the following paper are considered some simple systems of vortices in a compressible fluid. To show that such systems are of considerable importance it is sufficient to refer to the phenomena of cyclonic storms. It may be as well, however, to state that though the vortices are here treated as compressible, the circumstances are still so different from those found in nature that the results obtained could bear only a general resemblance at most to the phenomena of storms.

For brevity, the reader is supposed to understand and have in his hands for reference, Prof. Lamb's "Motion of Fluids," the symbols employed here having the same meaning, while the equations with suffix L refer to the equations so numbered in chapter VI. of that treatise.

When the fluid is limited the velocity must everywhere be tangential to the boundary. It is shown by Lamb how straight vortices parallel to one or more plane boundaries may be treated; and it is easily seen that a vortex of any shape in presence of an infinite plane boundary requires merely the introduction on the other side of the boundary of a vortex coinciding in position with the image of the real vortex as given by a plane mirror coinciding with the infinite plane. The direction of rotation in the imaginary vortex is such that the motion it causes tangential to the plane is in the same direction as that due to the real vortex. If the fluid be compressible it is only necessary that the imaginary fluid have at every point the same density as in the corresponding real point at the instant considered. If, for instance, xy be a boundary plane, vortices parallel to oz must be supposed to extend to infinity in both directions, and the density will be the same at any two points at equal distances from xy in a perpendicular to it.

Suppose, now, an infinite straight vortex parallel to oz, and of uniform section throughout, the density being some function of z, which we shall suppose not to change sign with z, then the motion

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is in two dimensions. For the strength of the vortex being throughout its length constant, we have only the functions N and P, of which the former is given by (33_L), or when the cross section is small by

$$N = -\frac{m}{\pi}\log r. \tag{1}$$

Also assuming the velocity everywhere parallel to xy, and denoting by p the pressure in the fluid, and by Z the component of the external forces parallel to oz, we have

$$Z - \frac{1}{\rho}\frac{dp}{dz} = 0.$$

Provided, then, the external forces are wholly parallel to oz, and Z is a given function of z, whether $p = k\rho$ or $= k\rho^\gamma$ where k and γ are constants, the above equation determines the relation between the density at the plane xy and at the distance z from that plane, which will exist supposing no velocity parallel to oz. If now σ be the cross section at time t, and ρ the density at any height, supposed uniform over the cross section, the equation of continuity is simply $\frac{\delta}{\delta t}(\sigma\rho) = 0$, where, as in what follows, δ denotes differentiation following the fluid; whence we see that $-\frac{1}{\rho}\frac{\delta\rho}{\delta t}$ is $= \frac{1}{\sigma}\frac{\delta\sigma}{\delta t}$, and so is independent of z.

Thus for this elementary column we get

$$P = \frac{1}{2\pi}\log r\,\frac{\delta\sigma}{\delta t}. \tag{2}$$

The velocity depending on P at a distance r from the axis of the column of varying density is radial, and is given by

$$\frac{dP}{dr} = \frac{1}{2\pi r}\frac{\delta\sigma}{\delta t} \tag{3}$$

Thus the fluid crossing a cylinder of radius r coaxial with the column of varying density, is simply $\frac{\delta\sigma}{\delta t}$ per unit of length in unit time. It follows that as much fluid leaves as enters the space between any two such cylindrical surfaces, and so the presence of a column of varying density has no direct tendency to cause variation in the density of the surrounding fluid.

Suppose we have a single thin straight filament parallel to oz, its vorticity being of strength m and its cross section at time t being σ; then the velocity at distance r from the axis is $\frac{dP}{dr}$ along and $\frac{dN}{dr}$ perpendicular to r. Thus, taking the axis of z along the axis of the

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filament, the component velocities parallel to rectangular axes of x and y are

$$u = -\frac{my}{\pi r^2} + \frac{1}{2\pi}\frac{\delta\sigma}{\delta t}\frac{x}{r^2} \tag{4},$$

$$v = \frac{mx}{\pi r^2} + \frac{1}{2\pi}\frac{\delta\sigma}{\delta t}\frac{y}{r^2} \tag{5}.$$

If the vortex, though of larger cross section, be circular and ρ and ζ be both functions only of the distance from the centre the same formulae will apply.

Unless $\frac{\delta\sigma}{\delta t}$ be constant the stream lines due to such a vortex vary in position with the time; if, however, $\frac{\delta\sigma}{\delta t}$ be constant they are equiangular spirals of the type

$$r = ae^{\frac{1}{2m}\frac{\delta\sigma}{\delta t}\phi} \tag{6},$$

denoting by ϕ an angle measured from some fixed plane through oz.

Let us next consider two thin vortices similar to the last—viz., the vortex m_1, σ_1 with its centre at the point (x_1, y_1), and the vortex m_2, σ_2 with its centre at (x_2, y_2). Then for the motion of the vortices we have

$$\frac{\delta x_2}{\delta t} = -\frac{m_1}{\pi r^2}(y_2 - y_1) + \frac{1}{2\pi}\frac{\delta\sigma_1}{\delta t}\frac{x_2 - x_1}{r^2} \tag{7},$$

$$\frac{\delta y_2}{\delta t} = \frac{m_1}{\pi r^2}(x_2 - x_1) + \frac{1}{2\pi}\frac{\delta\sigma_1}{\delta t}\frac{y_2 - y_1}{r^2} \tag{8},$$

$$\frac{\delta x_1}{\delta t} = -\frac{m_2}{\pi r^2}(y_1 - y_2) + \frac{1}{2\pi}\frac{\delta\sigma_2}{\delta t}\frac{x_1 - x_2}{r^2} \tag{9},$$

$$\frac{\delta y_1}{\delta t} = \frac{m_2}{\pi r^2}(x_1 - x_2) + \frac{1}{2\pi}\frac{\delta\sigma_2}{\delta t}\frac{y_1 - y_2}{r^2} \tag{10},$$

where $r^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$.

These equations would also apply approximately to the case of two circular vortices of larger cross section, provided r were great compared to the radius of either.

From (7) – (10) we get

$$(x_2 - x_1)\frac{\delta}{\delta t}(x_2 - x_1) + (y_2 - y_1)\frac{\delta}{\delta t}(y_2 - y_1) = \frac{1}{2\pi}\frac{\delta}{\delta t}(\sigma_1 + \sigma_2)$$

$$\therefore \quad r^2 = a^2 + \frac{1}{\pi}\{\sigma_1 + \sigma_2 - {}_0\sigma_1 - {}_0\sigma_2\} \tag{11},$$

if at the time $t = o$, $r = a$, $\sigma_1 = {}_0\sigma_1$, and $\sigma_2 = {}_0\sigma_2$.

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Again, from (7) – (10) it is easy to prove

$$\frac{\delta}{\delta t}\left(\frac{y_2-y_1}{x_2-x_1}\right)=\frac{m_1+m_2}{\pi(x_2-x_1)^2} \tag{12}.$$

Suppose, now, the line joining the vortices to be at time t inclined at an angle ϕ to its original direction, which is taken as axis of x; then the above equation becomes

$$\frac{1}{\sec^2\phi}\frac{\delta}{\delta t}(\tan\phi)=\frac{m_1+m_2}{\pi r^2},$$

whence

$$\phi=\frac{m_1+m_2}{\pi}\int_0^t\frac{dt}{a^2+(\sigma_1+\sigma_2-{}_0\sigma_1-{}_0\sigma_2)t/\pi} \tag{13.}$$

If, then, the law of variation of σ_1 and σ_2 be given, the distance of the vortices and the direction of the line joining them follow at once from (11) and (13). Suppose, for instance,

$$\left.\begin{aligned}\sigma_1&={}_0\sigma_1(1+\gamma_1 t)\\ \sigma_2&={}_0\sigma_2(1+\gamma_2 t)\end{aligned}\right\} \tag{14},$$

where γ_1 and γ_2 are constants; then, if $\dfrac{2(m_1+m_2)}{{}_0\sigma_1\gamma_1+{}_0\sigma_2\gamma_2}$ be denoted by B, we get

$$r^2=a^2\left\{1+\frac{2(m_1+m_2)}{\pi a^2\mathrm{B}}t\right\} \tag{15},$$

and

$$\phi=\frac{\mathrm{B}}{2}\log\left\{1+2\frac{(m_1+m_2)}{\pi a^2\mathrm{B}}t\right\} \tag{16},$$

whence

$$r^2=a^2e^{2\phi/\mathrm{B}} \tag{17}.$$

In this case the path of one vortex relative to the other is an equiangular spiral, and the time of the n^{th} complete revolution of the line joining them is

$$t_n=\frac{\pi a^2\mathrm{B}}{2(m_1+m_2)}(e^{4\pi/\mathrm{B}}-1)e^{4(n-1)\pi/\mathrm{B}} \tag{18}.$$

To find the absolute motion, let X, Y be the co-ordinates of the centre of inertia of the masses m_1 at (x_1, y_1) and m_2 at (x_2, y_2); so that

$$\begin{aligned}(m_1+m_2)\mathrm{X}&=m_1x_1+m_2x_2,\\ (m_1+m_2)\mathrm{Y}&=m_1y_1+m_2y_2.\end{aligned}$$

Then from (7)—(10) we get

$$\left.\begin{aligned}(m_1+m_2)\frac{\delta\mathrm{X}}{\delta t}&=\frac{1}{2\pi}\frac{x_2-x_1}{r^2}\left(m_2\frac{\delta\sigma_1}{\delta t}-m_1\frac{\delta\sigma_2}{\delta t}\right)\\ (m_1+m_2)\frac{\delta\mathrm{Y}}{\delta t}&+\frac{1}{2\pi}\frac{y_2-y_1}{r^2}\left(m_2\frac{\delta\sigma_1}{\delta t}-m_1\frac{\delta\sigma_2}{\delta t}\right)\end{aligned}\right\} \tag{19.}$$

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If $m_2 \frac{\delta\sigma_1}{\delta t} - m_1 \frac{\delta\sigma_2}{\delta t}$ vanish, which includes the more special case when σ_1 and σ_2 remain constant, we have

$$X = \text{constant}, \quad Y = \text{constant}.$$

In the general case we could not integrate the expressions for X and Y; but if the relations (14) hold, we have

$$(m_1 + m_2) \frac{\delta X}{\delta t} = \frac{A}{r}\cos\phi,$$

$$(m_1 + m_2) \frac{\delta Y}{\delta t} = \frac{A}{r}\sin\phi,$$

where $$A = \frac{m_2 {}_0\sigma_1\gamma_1 - m_1 {}_0\sigma_2\gamma_2}{2\pi}.$$

By means of (15) we may regard r as the variable instead of t, and so employing (17) get

$$(m_1 + m_2) \frac{dX}{dr} \frac{r\delta r}{\delta t} = A \cos\left(B \log \frac{r}{a}\right);$$

but $$\frac{\delta r}{r\delta t} = \frac{m_1 + m_2}{\pi B},$$

$\therefore$ $$X = \int C \cos\left(B \log \frac{r}{a}\right) dr,$$

where $$C = \frac{\pi AB}{(m_1 + m_2)^2}.$$

Integrating this by parts, and supposing when $t = o$ and $r = a$ that $X = o = Y$, we get

$$X = \frac{C}{1 + B^2}\left[r\cos\left(B\log\frac{r}{a}\right) - a + Br\sin\left(B\log\frac{r}{a}\right)\right\}\Big] \qquad (20).$$

Similarly we find

$$Y = \frac{C}{1 + B^2}\left[r\sin\left(B\log\frac{r}{a}\right) + B\left\{a - r\cos\left(B\log\frac{r}{a}\right)\right\}\right] \qquad (21).$$

Since r and ϕ have been already determined the motion is in this case completely investigated.

We see from (11) that in every case the vortices approach to or recede from one another according as the sum of their cross sections is diminishing or increasing, the rate of approach being independent either of the directions or magnitudes of their vorticities. Since σ_1 and σ_2 must be positive, the distance of the vortices can never be less than $\sqrt{a^2 - \frac{1}{\pi}({}_0\sigma_1 + {}_0\sigma_2)}$; and as we have practically assumed the distance a great compared to the radius of either vortex,

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it follows that the distance can experience only a comparatively small diminution. In particular, in applying the results deduced on the hypothesis (14) we must limit t so that $1+\gamma_1 t$ and $1+\gamma_2 t$ are both positive.

A special interest as will be seen attaches to the case $m_2 = -m_1$. The equation (11) still holds, but in place of (13) we have $\phi = 0$ and so $y_2 = y_1$, the vortices being supposed initially to have the centres of their bases in the axis of x. It follows at once from the equations that

$$(x_2 - x_1)^2 = r^2 = a^2 + \frac{1}{\pi}(\sigma_1 + \sigma_2 - {}_0\sigma_1 - {}_0\sigma_2) \tag{22}.$$

while

$$y_2 = y_1 = \frac{m_1}{\pi}\int \frac{\delta t}{\sqrt{a^2 + \frac{1}{\pi}(\sigma_1 + \sigma_2 - {}_0\sigma_1 - {}_0\sigma_2)}} \tag{23}.$$

If we suppose $\frac{\delta\sigma_1}{\delta t} = \frac{\delta\sigma_2}{\delta t}$ then the equations give $\frac{\delta x_2}{\delta t} = -\frac{\delta x_1}{\delta t}$, or the vortices have equal but opposite velocities in the line joining them; and if the point half-way between them be taken as origin, we have

$$x_2 = -x_1 = \tfrac{1}{2}a\sqrt{1 + 2(\sigma_1 - {}_0\sigma_1)/\pi a^2} \tag{24}.$$

$$y_2 = y_1 = \frac{m_1}{\pi a}\int \frac{\delta t}{\sqrt{1 + 2(\sigma_1 - {}_0\sigma_1)/\pi a^2}} \tag{25}.$$

This answers to the case when a single vortex exists parallel to a boundary plane, taken as that of yz, the original distance being $\frac{a}{2}$. If, again, while $m_2 = -m_1$ we have $\sigma_1 + \sigma_2$ constant then $r^2 = a^2$; and, taking the original position of the middle point of the line joining the vortices as origin, we get

$$x_2 = \frac{a}{2} + \frac{1}{2\pi a}(\sigma_1 - {}_0\sigma_1),$$

$$x_1 = -\frac{a}{2} + \frac{1}{2\pi a}(\sigma_1 - {}_0\sigma_1),$$

and

$$y_2 = y_1 = \frac{m_1}{\pi a}t.$$

If, while still supposing $m_2 = -m_1$ we suppose the relations (14) to hold, it is easy to prove that the vortices move along straight lines; viz. m_2 along the line $y = \frac{2m_1}{{}_0\sigma_1\gamma_1}\left(x - \frac{a}{2}\right)$ and m_1 along $y + \frac{2m_1}{{}_0\sigma_2\gamma_2} \times \left(x + \frac{a}{2}\right) = 0$, the origin being at the initial position of the middle point of the line joining the vortices.

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The methods here employed will also give the motion when any number of vortices exist, but the difficulty of solving the equations increases rapidly with the number of vortices. As an example suppose the planes of zx and zy to form two infinite boundaries at right angles. Let a straight vortex filament of strength m parallel to oz have at time t the cross section σ, and let (x, y) be the centre of the cross section by the plane xy. Then the planes may be supposed non-existent and the fluid infinite if we introduce three new vortices all of cross section σ. One of these has its centre at $(-x, -y)$ and is of strength m, the other two are each of strength $-m$ and have their centres at $(x, -y)$ and $(-x, y)$ respectively. The velocities of the original vortex are

$$\left.\begin{aligned} u &= \frac{m}{2\pi}\frac{x^2}{y(x^2+y^2)} + \frac{1}{4\pi}\frac{\delta\sigma}{\delta t}\frac{2x^2+y^2}{x(x^2+y^2)} \\ v &= -\frac{m}{2\pi}\frac{y^2}{x(x^2+y^2)} + \frac{1}{4\pi}\frac{\delta\sigma}{\delta t}\frac{x^2+2y^2}{y(x^2+y^2)} \end{aligned}\right\} \qquad (26).$$

To determine completely the path described would in the general case be difficult. If, however, $\frac{\delta\sigma}{\delta t} = \text{constant} = km$, and (r, ϕ) denote the polar co-ordinates answering to (x, y), it is not difficult to prove that the path of the vortex is the curve

$$\left(\frac{r}{r_0}\right)^{1+k^2}\left\{\frac{\sin 2\phi - k\cos 2\phi}{\sin 2\phi_0 - k\cos 2\phi_0}\right\}^{1+3k^2/4} = e^{k(\phi-\phi_0)/2} \qquad (27),*$$

where r_0, ϕ_0 are the original values of r, ϕ.

In any application of the preceding results to the case of vortices in the earth's atmosphere it must be observed that the vortices are here supposed to exist in a fluid limited only by infinite planes and not revolving as a whole about an axis. In the case of the earth, the vortices are at distances apart comparable with the earth's radius, and the vortex motion is directly influenced by the earth's rotation; the effect also of the rotation in modifying the atmospheric density in different latitudes is of great importance. Further, the motion considered here is only in two dimensions; while in cyclonic storms the velocity and even the direction of the wind seem often to vary at

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* For special case of incompressible fluid see a paper by Professor Greenhill in the "Quarterly Journal of Mathematics," Vol. XV.

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different altitudes above the ground, and some observers assert that in the centre of the disturbance ascending vertical currents often exist. In connection with this point it may be of interest to refer to Buys Ballot's law for cyclonic storms. In these there is a central area where the barometer is low and the wind blows round this area. According to the law in question the wind does not blow perpendicularly to the line joining the observer to the point where the barometer is lowest, but is more or less directed towards the centre of the depression. Now, in accordance with the results we have obtained, if the motion were in two dimensions this law would be true only if the section of the vortex were contracting, in which case the density would be increasing and the barometer rising at the centre of the depression. Further, the magnitude of the radial velocity would be proportional to the rate of variation with the time in the height of the barometer. If the barometer were falling throughout the area of the disturbance the direction of the wind would be on the whole outwards from the centre. Thus, supposing Buys Ballot's law well founded, we must conclude either that vertical currents do exist in the centre of cyclonic storms, or else that cyclonic depressions fill up in much less time than they take to form. It should also be noticed that the rate of fluctuation of the barometer at any one station affords no clue to the law of fluctuation of the density at the centre of the disturbance. A rapid fall, for instance, might mean merely that the storm had a rapid motion of translation, or that the density diminished rapidly in approaching the centre of the depression.

A Theorem in Algebra.

By J. L. Mackenzie.

If we have given two equations $\phi(x) = 0$ and $\psi(y) = 0$, it is possible to express in the form of a determinant the equation whose roots are $f(x, y)$, where f is any given rational integral function.

Let α_r, β_r, be the sums of the r^{th} powers of the roots of the given equations, and s_r of the required equation. Then

$$s_r = \{f(a_1, b_1)\}^r + \{f(a_2, b_1)\}^r + \ldots\ldots$$
$$+ \{f(a_1, b_2)\}^r + \{f(a_2, b_2)\}^r + \ldots\ldots \text{ \&c.}$$

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Thus to any term $\lambda x^m y^n$ in the expansion of $\{f(x, y)\}^r$, there will correspond in s_r the sum

$$\lambda a_1^m b_1^n + \lambda a_2^m b_1^n + \ldots\ldots\ldots + \lambda a_1^m b_2^n + \lambda a_2^m b_2^n + \ldots\ldots \text{ \&c.}$$
$$= \lambda a_m \beta_n.$$

Hence s_r may be found by expanding $\{f(x, y)\}^r$, and substituting a_m, β_n, for x^m, y^n in every term of the expansion.

It is not difficult to extend this to many cases where f is not a rational nor an integral function, and where there are more than two equations $\phi(x)$, $\psi(y)$, &c.

When s_r is known, the required equation is obtained in the form of a determinant by eliminating $1, p_1, p_2 \ldots\ldots\ldots p_n$ between

$$u^n + p_1 u^{n-1} + p_2 u^{n-2} + \ldots\ldots\ldots + p_n = 0$$

and any n of Newton's equations

$$s_1 + p_1 = 0$$
$$s_2 + p_1 s_1 + 2p_2 = 0 \qquad \text{\&c.}$$

Taking the first n of these we get

$$\begin{vmatrix} u^n & u^{n-1} & u^{n-2} \ldots\ldots 1 \\ s_1 & 1 & 0 \quad \ldots\ldots 0 \\ s_2 & s_1 & 2 \quad \ldots\ldots 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ s_n & s_{n-1} & s_{n-2} \ldots\ldots n \end{vmatrix} = 0.$$

It is evident that the only extraneous factor in this equation is $n\,!$

The calculations required for finding and reducing this determinant are usually very laborious; but I have applied this method to calculate the equations of certain loci derived from two conics. Write the equations of the conics in the form

$$u_2 + u_1 + u_0 = 0,$$
$$v_2 + v_1 + v_0 = 0;$$

or, in polar co-ordinates,

$$r_1^2 - P_1 r_1 + P_2 = 0,$$
$$r_2^2 - Q_1 r_2 + Q_2 = 0,$$

where $$P_1 = -\frac{u_1 r}{u_2},\ P_2 = \frac{u_0 r_2}{u_2},$$

with similar values for Q_1 and Q_2.

Then

$$\alpha_1 = P_1,$$
$$\alpha_2 = P_1^2 - 2P_2,$$
$$\alpha_3 = P_1^3 - 3P_1 P_2 \qquad \text{\&c.},$$

and similarly for β_r.

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(1.) If a line through the origin cut the conic U in A, and V in B, to find the equation of the locus of a point P which divides AB in the ratio $\lambda : \mu$, $(\lambda + \mu = 1)$.

Here the required equation is

$$\begin{vmatrix} r^4 & r^3 & r^2 & r & 1 \\ 2(\lambda a_1 + \mu\beta_1), & 1 & 0 & 0 & 0 \\ 2(\lambda^2 a_2 + \lambda\mu a_1\beta_1 + \mu^2\beta_2), & s_1 & 2 & 0 & 0 \\ 2\lambda^3 a_3 + 3\lambda^2\mu a_2\beta_1 + 3\lambda\mu^2 a_1\beta_2 + 2\mu^3\beta_3, & s_2 & s_1 & 3 & 0 \\ 2\lambda^4 a_4 + 4\lambda^3\mu a_3\beta_1 + 6\lambda^2\mu^2 a_2\beta_2 + 4\lambda\mu^3 a_1\beta_3 + 2\lambda^4\beta_4, & s_3 & s_2 & s_1 & 4 \end{vmatrix}$$

This, when finally reduced, gives

$$\begin{aligned} & u_2^2(v_2 + \mu v_1 + \mu^2 v_0)^2 + \lambda^2 u_1^2 v_2(v_2 + \mu v_1 + \mu^2 v_0) \\ & + \lambda^2 u_0 u_2(2v_2^2 + 2\mu v_1 v_2 - 2\mu^2 v_0 v_2 + \mu^2 v_1^2) \\ & + \lambda u_1 u_2(\mu v_1 + 2v_2)(v_2 + \mu v_1 + \mu^2 v_0) \\ & + \lambda^2 u_0 u_1 v_2(\mu v_1 + 2v_2) \\ & + \lambda^4 u_0^2 v_2^2 = 0. \end{aligned}$$

(2.) If $OP^2 = OA.\ OB$, we get by the same method,

$$\begin{aligned} & u_2^2 v_2^2(u_2 v_2 - u_0 v_0)^2 - u_1 v_1 u_2^2 v_2^2(u_2 v_2 + u_0 v_0) \\ & + (u_2^2 v_2^2 - 4u_0 v_0 u_2 v_2)(u_1^2 v_0 v_2 + u_0 u_2 v_1^2) \\ & + u_1^4 v_0^2 v_2^2 + u_0^2 u_2^2 v_1^4 = 0. \end{aligned}$$

(3.) Finally, if P be the harmonic conjugate of O with respect to A and B, we get for the equation to the locus of P,

$$\begin{aligned} & (u_2 v_0 - u_0 v_2)^2 + (u_2 v_1 + u_1 v_2)(u_1 v_0 + u_0 v_1) \\ & + (4u_2 v_0 + u_1 v_1 + 4u_0 v_2)(u_1 v_0 + u_0 v_1) \\ & + 4(u_1 v_0 + u_0 v_1)^2 + 4u_0 v_0 u_1 v_1 \\ & + 8u_0 v_0(u_2 v_0 + u_0 v_2) \\ & + 16u_0 v_0(u_1 v_0 + u_0 v_1) \\ & + 16u_0^2 v_0^2 = 0. \end{aligned}$$

A Trigonometrical Note.

By Dr J. S. Mackay.

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Fifth Meeting, March 11th, 1887.

Dr George Thom, President, in the Chair.

Historical Notes on a Geometrical Theorem and its Developments.

[18th century.]

By J. S. Mackay, LL.D.

The theorem is—

The distance between the circumscribed and the inscribed centres of a triangle is a mean proportional between the circumscribed radius and its excess above the inscribed diameter.

Or, in other words,

The potency of the inscribed centre of a triangle with respect to the circumscribed circle is equal to twice the rectangle under the inscribed and the circumscribed radii.

The notes are arranged as far as possible in chronological order, under the names of the various geometers who have turned their attention to the question. The first paper is given nearly in full, partly because the author of it seems to be totally unknown outside of the United Kingdom, and partly because the periodical in which the paper appeared is so rare and so difficult to get that it seems a sort of mockery to refer any one to it. Some few changes have been made on the lettering of the figures, and on the notation for certain lines.

WILLIAM CHAPPLE (1746).

The *Miscellanea Curiosa Mathematica* was begun, under the editorship of Francis Holliday, in the year 1745. It was to be published quarterly, but the fact that the first volume contained only nine numbers, and that the dedication prefixed to it is dated March 25, 1749, seems to show that it cannot have appeared at the regular intervals intended. Probably it is not rash to suppose that

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the fourth number was published in 1746. That number opens with "An Essay on the properties of triangles inscribed in, and circumscribed about two given circles; by Mr William Chapple" (pp. 117-124). Before entering upon his subject Chapple remarks:—

The following enquiry into the properties of triangles inscribed in, and circumscribed about given circles, has led me to the discovery of some things relating to them, which I presume have not been hitherto taken notice of, having not met with them in any author; though an ingenious correspondent of mine, in the isle of Scilly, to whom I communicated some of the propositions hereinafter demonstrated, informs me that he had begun to consider it some years ago, but did not go through with it; however, I must acknowledge that a query of his to me, relating thereto, gave me the first hint, and induced me to pursue the subject with more attention than perhaps otherwise I might have done.

PROPOSITION 1.

The areas of all triangles, circumscribed about the same circle, are as their perimeters.

For the areas of all are equal to their perimeters multiplied into half the radius.

PROPOSITION 2.

The areas of all triangles, inscribed in the same circle, are as the products of their sides.

For the areas of all are equal to the product of their sides divided by twice the diameter of their circumscribing circle.

COROLLARY.

The areas of all triangles, inscribed and circumscribed in and about the same circles, are not only as their perimeters, but also as the products of their sides.

PROPOSITION 3.

To inscribe and circumscribe a triangle in and about two concentric circles, the radius of the greater circle must be double the radius of the lesser, and the triangle will always be equilateral.

[This is so easily established that I omit Chapple's demonstration.]

PROPOSITION 4.

To inscribe and circumscribe a triangle in and about two eccentric circles, the radius of the lesser circle must be less than half the radius of the greater circle.

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The area of any triangle is equal to the product of the radius of its inscribed circle into half the perimeter; therefore, if this proposition be true, the area of any triangle doth not exceed the product of half the radius of its circumscribed circle into half the perimeter. Now a triangle whose area is equal to the product of half the radius of its circumscribed circle into half the perimeter is equilateral, by Proposition 3. So that, if this proposition be true, the greatest triangle that can be inscribed in any circle will be equilateral; which we come now to demonstrate.

[This also is so easily established that I omit Chapple's demonstration.]

PROPOSITION 5.

An infinite number of triangles may be drawn, which shall inscribe and circumscribe the same two circles; provided their diameters, with respect to each other, be limited as in the two last propositions.

For, put x, y, and z equal to the sides of any triangle circumscribed about a circle whose radius is r and inscribed in a circle whose radius is R. Then $xyz/4R =$ the area of the triangle,

and $(x+y+z)\ r/2 =$ the area.

Therefore $$xyz = 2\ Rr\ (x+y+z).$$

Hence it is plain that if the sides of the triangle were required to be found by the given diameters, the question would be capable of innumerable answers; for one of the sides at least is unlimited, and may be put equal to anything at pleasure, that doth not exceed the longest line that can be drawn within the great circle as a tangent to the lesser. And hence it appears, that if the distance of the given circles be so fixed, as that any one triangle may be inscribed and circumscribed, innumerable others may be inscribed and circumscribed in and about the same circles.

PROPOSITION 6.

The nearest distance of the peripheries of two given circles, or, which amounts to the same, the distance of their centres, in order to render it possible to inscribe and circumscribe triangles, is fixed and will be always the same.

Let two circles be so situated as that a triangle ABC can be circumscribed and inscribed; then innumerable others may be inscribed and circumscribed; but it is evident by inspection that if the lesser circle be anyhow removed from its place, the sides of those triangles cannot be tangents to it; and therefore the situation of the

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two circles must be the same for them all. But if it be suspected that their situation may be altered, and yet other innumerable triangles may be circumscribed and inscribed (as the contrary is not yet made appear) let it be considered, that the lesser circle must, in its removal, move along on some one or other of the sides of the innumerable triangles that might be inscribed and circumscribed at its *first situation*, and then wherever it stops (unless it be at the same distance from the periphery of the great circle on the other side of the centre, which is in effect the same situation) the like inconveniency will follow. For it is well known that triangles circumscribed about equal circles and having one common base, will continually increase their altitude, the further the point of contact in the said base is removed from the middle thereof, till at length the two lines drawn from the extremities thereof become parallel to or diverging from each other, and the altitude of the triangle becomes infinite; consequently (when the two circles are eccentric) the vertices of *only two* of the circumscribing triangles that can be erected on the base AB can be at the periphery of the circumscribing circle; and in either of these the distance of the peripheries of the greater and lesser circles must be the same, and the triangles the same, having the same base and altitude. So that the proposition is abundantly proved; and hence we have the following

COROLLARY.

If one side of a triangle, inscribed in and circumscribed about two given circles, be given, the other two sides are thereby limited, and may from thence be found.

[For the method of finding them Chapple refers to the answer to a question he had proposed for solution in the *Miscellanea*. The question and answer will be given later on.]

PROPOSITION 7.

Of the innumerable triangles that may be inscribed and circumscribed in and about two given eccentric circles, two must of course be isosceles, the vertices of which will be in the common diameter of those circles, which will cut their bases at right angles; now the content of that isosceles triangle which hath the least base and greatest altitude will be the greatest, *and that of the other the* least *of all the triangles that can be inscribed and circumscribed in the given circles.*

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See figure 71.

Let S and I be the centres of the greater and less circles, and let the two isosceles triangles ABC, DEF, whose vertices are at the ends of the common diameter AD, be inscribed in the one circle and circumscribed about the other; ABC is the greatest and DEF the least of the inscribed and circumscribed triangles.

Let AD meet BC at H; then H is the middle point of BC.
Find K the middle point of AB, and join CK, meeting AH in G.
Then $AG = \frac{2}{3}AH$, $CG = \frac{2}{3}CK$, $\triangle AGC = \frac{1}{3}\triangle ABC$.

Now innumerable lines may be drawn in the semicircle AED, all of which shall be tangents to the lesser circle, and sides of inscribed and circumscribed triangles, and such lines must always be less than AC, because farther removed from S, the centre of the great circle, and will continually decrease according as their extremities are farther removed from A, so that ED will be the least of all of them. Let LN be any one of those lines, and let the circumscribing and inscribed triangle LMN be drawn; also the lines LP and NQ bisecting MN and ML, and crossing each other in O, the centre of gravity of this triangle; so that $\triangle LON = \frac{1}{3}\triangle LMN$.

Because an angle of any triangle is greater or less according as it is nearer to or farther from the centre of the inscribed circle, therefore $\angle L$ is greater than $\angle A$;
therefore LP is less than AH;
therefore $\frac{2}{3}$ LP is less than $\frac{2}{3}$ AH, that is, LO is less than AG.
Similarly NO is less than CG.
Now LN, being also less than AC, the area of $\triangle LON$ is less than the area of $\triangle AGC$;
therefore $\triangle ABC$ is greater than $\triangle LMN$.

Hence it plainly appears that the area of any circumscribing and inscribed triangle which hath one angle anywhere between A and E is less than the area of the isosceles triangle ABC, and greater than the isosceles triangle DEF. And that this takes in all the cases that can possibly happen will be evident to any one that considers that if the angle L be between A and E, the angle N will be somewhere between C and D, and the angle M somewhere between B and F; so that this takes in one-half of the periphery; and the arc AF being equal to AE, and BF equal to CE, it is but changing sides with the circle, and we have the same set of triangles again; for wherever the triangle be, if it be not isosceles, we may have another

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on the other side of the centre of the great circle, which shall be equal to it.

From the three foregoing propositions we may deduce the following

COROLLARY.

If the distance of the centres of the two given circles be not so fixed as that an isosceles triangle can be inscribed and circumscribed, no triangle whatsoever can be inscribed and circumscribed.

And this suggests an easy method of finding the distance of the centres of the said given circles, and consequently the nearest distance of their peripheries; for here we may always consider the triangle as isosceles, and then the distance is found as follows.

See figure 71.

Put $AH = x$, $CH = y$, $SA = R$, $IH = r$;

then $DH = 2R - x$, and $CH = y = \sqrt{2Rx - x^2}$;

also $AC = \sqrt{x^2 + y^2}$.

Now, sine of $\angle CAH = \frac{CH}{AC} = \frac{y}{\sqrt{x^2 + y^2}}$,

and sine of $\angle CAH = \frac{IT}{IA} = \frac{r}{x - r}$;

therefore $$\frac{y}{\sqrt{x^2 + y^2}} = \frac{r}{x - r}.$$

In this equation substitute for y the value $\sqrt{2Rx - x^2}$, and complete the solution; then

$x - R - r = \sqrt{R^2 - 2Rr}$ = the distance of the centres;

therefore $x = AH = R + r + \sqrt{R^2 - 2Rr}$

and $DH = R - r - \sqrt{R^2 - 2Rr}$, the nearest distance of the peripheries.

Note—That R must be equal to or greater than $2r$, or else the root is impossible, which is another proof of the fourth proposition.

The question proposed for solution by Chapple (Vol. I., p. 143) and answered by him (Vol. I., pp. 171-3) was—

Let the two diameters of the circumscribing and inscribed circles of a triangle be 94 *and* 40, *and one of its sides* 90; *quere the other two sides.*

Denoting the radius of the inscribed circle by r, that of the circumscribed by R, the given side by a, and the distance between the

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middle of the given side and the point of inscribed contact on that side by e, Chapple deduces the expression

$$\sqrt{2Rr + e^2 + \frac{4R^2r^2}{a^2}} + \frac{2Rr}{a} \pm e$$

for the two unknown sides, the affirmative value of e being taken for the greater, and the negative for the less.

Another question proposed by Chapple (Vol. I., p. 185) and answered by him (Vol. I., p. 196) was—

Given the hypotenuse of a right-angled triangle equal to 100, *and the nearest distance of the right angle from the periphery of the inscribed circle equal to* 8; *required the legs and area of the triangle by a simple equation, with a general theorem for questions of this nature.*

The general theorem given by Chapple at the end of his solution for finding the legs of right-angled triangles is

$$R + r \pm e = \left.\begin{matrix}\text{greater}\\ \text{lesser}\end{matrix}\right\} \text{ leg.}$$

He adds a corollary: The sum of the two legs of any right-angled triangle is equal to the sum of the diameters of the inscribed and circumscribing circles.

In order to draw attention to his paper in the *Miscellanea Curiosa Mathematica*, Chapple proposed the following as the prize question in the *Ladies' Diary* for 1746:—

A gentleman has a circular garden whose diameter is 310 *yards, in which is contained a circular pond whose diameter is* 100 *yards, so situated in respect of each other that their peripheries will inscribe and circumscribe an infinite number of triangles (*i.e., *whose sides shall be tangents to the pond, and angles in the fence of the garden). He being disposed to make enclosures for different uses, and farther ornaments on his scheme begun, in order thereto applies himself to the artists of Great Britain for the dimensions of the greatest and least triangles that can be inscribed and circumscribed as aforesaid? and the nearest distance of the peripheries of the garden and pond? and for a demonstration of the truth of the pond's situation?*

Robert Heath (1747).

Chapple's prize question was answered somewhat unsatisfactorily next year by Robert Heath, who was then editor of the *Ladies' Diary*. At the end of his solution he says:—This property of draw-

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ing triangles about circles I discovered some years ago, as may be seen in the *Monthly Oracle;* though the proposer has greatly deserved in a long account of it from Scilly, printed in a book called the *Quarterly Miscellanea Curiosa.*

Landen, who will be mentioned later on, seems also to have answered the question. He gave in an inconvenient form the expression

$$R+r\pm\sqrt{R^2-2Rr}$$

for the altitudes of the two isosceles triangles, and a construction for finding the centre of the pond.

John Turner (1748).

Another mathematical periodical, *The Mathematician* (London, 1751), edited, according to the title-page, "By a Society of Gentlemen," according to T. S. Davies by Turner, made its first appearance in 1745. Six numbers only were published, probably annually, and in the fourth of these Turner, who had been a frequent contributor to the *Miscellanea Curiosa Mathematica*, and who had answered in it the first of the questions here mentioned as proposed by Chapple, reproposed it (p. 256) in this form :—

One side of a triangle, together with the radii of its circumscribing and inscribed circles being given, to construct the triangle geometrically.

From his own solution given in the next number (pp. 311-313), I extract the construction, which is as simple as possible. His demonstration is not simple enough to be worth reproducing.

See figure 72.

Upon any point S, with an interval equal to the given radius of the circumscribing circle, let the circle BACV be described, in which apply the right line BC equal to the given side of the triangle. Bisect BC with the indefinite perpendicular UV ; in which take HT equal to the radius of the inscribed circle. Through T draw a parallel to BC, and upon V the point where UV intersects the circle, and with the distance VB, describe an arch cutting the parallel to BC in I ; then, if through I the line VIA be drawn meeting the circle in A, and BA CA be joined, ABC will be the triangle required.

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JOHN LANDEN (1755).

In his *Mathematical Lucubrations* (London, 1755) Landen devotes Part I. pp. 1–24 to "an investigation of a remarkable property of triangles described in a certain manner about a circle or an ellipsis." The investigation is given in a proposition which is subdivided into three cases with six figures, and followed by eight corollaries and the solution of twenty examples. Two cases and two figures only are reproduced here, these being all that are really required.

PROPOSITION.

See figures 74, 75.

The two circles ABM, DEF, whose centres are O and I respectively, being given in magnitude and position, let any given chord AB in the circle ABM touch the circle DEF at F; and from the extremities of that chord let two other tangents, AC BC, be drawn to the circle DEF, touching it at E and D, and intersecting each other at C: it is proposed to find the radius AS of the circle ABU circumscribing the triangle ABC.

Bisecting AB by a right line HM at right angles thereto, that line will pass through O, the centre of the given circle ABM, and also through S, the centre of the circumscribing circle ABU. Draw ID, IE, IF to the points of contact D, E, F; join CI, and draw IT parallel to AB intersecting HM in T. Join AU.

Let $AO = R$, $IF = r$, $IO = d$, $AH = b$, $AS = x$;

then $OH = \pm \sqrt{R^2 - b^2}$, $OT = \pm \sqrt{R^2 - b^2} - r$,

$$IT = FH = \sqrt{d^2 - R^2 + b^2 - r^2 \pm 2r\sqrt{R^2 - b^2}},$$

$$AF = AE = b + \sqrt{d^2 - R^2 + b^2 - r^2 \pm 2r\sqrt{R^2 - b^2}}.$$

$$BF = BD = b \sim \sqrt{d^2 - R^2 + b^2 - r^2 \pm 2r\sqrt{R^2 - b^2}},$$

Case 1. When the circle DEF falls within the triangle ABC.

$$EF = \pm \sqrt{x^2 - b^2},\ GF = x \pm \sqrt{x^2 - b^2}.$$

From the similar triangles AHU, IEC, an expression is found for CE or CD, which being added to the values of AE and BD found above, will give the values of AC and BC. Thence is obtained the value of AC + BC + AB, which being multiplied by $\frac{1}{2}r$ gives the area of triangle ABC.

Another expression for the area of triangle ABC is got by con-

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sidering it as equal to half the rectangle contained by AC, BC and the sine of angle ACB, which is b/x.

If these expressions be equated to each other, and the equation solved, the value of x or AS will be found to be

$$x = \frac{R^2 - d^2 \mp 2r\sqrt{R^2 - b^2}}{4r} + \frac{b^2 r}{R^2 - d^2 \mp 2r\sqrt{R^2 - b^2}}.$$

Landen remarks that if d^2 be equal to $R^2 - 2rR$, whatever b may be, x will be equal to R.

Case 2. When the circle DEF falls without the triangle ABC, the method of procedure is the same as before, the only changes being that GF is now $\pm\sqrt{x^2 - b^2} - x$, and that the area of triangle ABC is obtained by multiplying AC + BC − AB by $\frac{1}{2}r$. In this case the value of x or AS is found to be

$$x = \frac{\pm 2r\sqrt{R^2 - b^2} + d^2 - R^2}{4r} + \frac{b^2 r}{\pm 2r\sqrt{R^2 - b^2} + d^2 - R^2}.$$

Landen remarks that if in this case d^2 be equal to $R^2 + 2rR$, whatever b may be, x will be equal to R. He adds that the sign proper to $\sqrt{R^2 - b^2}$ or to $\sqrt{x^2 - b^2}$ is the upper or lower one, according as O and I, or according as S and I are on the same or contrary sides of AB.

Cor. 1. It follows from what has been said that, d being equal to $\sqrt{R^2 - 2rR}$ or $\sqrt{R^2 + 2rR}$, whatever b may be, S will fall in O, and the circle circumscribing the triangle always coincide with the given circle ABM; *a thing very remarkable!*

For this to happen and the circle DEF fall within the triangle, it is obvious R must not be less than $2r$. But, that circle falling without the triangle, the same thing may happen though R be less than $2r$, so that R be greater than $\frac{1}{4}r$.

Cor. 2. "In orthographic projections, circles having the same inclination to the plane of projection being projected into similar ellipses, and any tangent of a circle projected into an ellipsis being likewise, when projected therewith, a tangent to that ellipsis: it follows that, if, within or without any ellipsis whose transverse axis is T, a second concentric similar ellipsis be described with its transverse axis t in the same direction with T; and a third ellipsis be described, similar to the other two, with its centre anywhere in

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the periphery of the second ellipsis, and having its transverse axis equal to $\frac{T^2 \sim t^2}{2T}$, and parallel to the transverse axes of the other ellipses; any tangent being drawn to this third ellipsis and continued both ways till it intersects the periphery of the first ellipsis in two points, and two other tangents being drawn to the same third ellipsis from those points of intersection, the locus where these last tangents continued intersect each other will always be in the periphery of the first ellipsis.

"The drawing of tangents in that manner will be impossible unless t be less than $3T$."

Cor. 3. Here and in the examples, by *the* escribed circle is understood the circle touching the base and the other two sides produced; its radius is denoted by ρ, and the distance of its centre from the circumscribed centre by δ.

The corollary gives, in terms of the base, the circumscribed radius and the inscribed or escribed radius, twenty-six expressions for the other two sides of the triangle, the sum, difference, and product of those sides, the perimeter of the triangle, the area, the perpendicular from the vertex to the base, the distances of the circumscribed centre from the inscribed and escribed centres, and the distances of these latter centres from the vertex. The only expression important enough to quote is

$$\delta = \sqrt{R^2 + 2\rho R}.$$

Cor. 4. This consists of twenty-two expressions for the other two sides of the triangle, the sum, difference, and product of those sides, the perimeter of the triangle, the area, the perpendicular from the vertex to the base, and the distances of the circumscribed centre from the inscribed and escribed centres in terms of the base, the inscribed or escribed radius, and the sine of the vertical angle or the cotangent of half that angle.

Cor. 5. Here the triangle is supposed to be right-angled, the base becoming the hypotenuse. Seventeen expressions in terms of the circumscribed radius and the inscribed or escribed radius are given for the other two sides, their sum, difference, product, the perimeter,

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and the perpendicular from the vertex to the hypotenuse. The following are worth noting:

The product of the sides about the right angle

$$= 4rR + 2r^2 = 2r\rho = -4\rho R + 2\rho^2;$$

the perpendicular from the vertex to the hypotenuse

$$= 2r + \frac{r^2}{R} = \frac{r\rho}{R} = -2\rho + \frac{\rho^2}{R};$$

the perimeter $= 2\rho$.

The sixth, seventh, and eighth corollaries are of little interest.

The following is a statement of the examples with whose solution Landen's essay concludes:—

1. Let the radii of the circles circumscribed about, and inscribed in a triangle be 5 and 2·2 respectively, and one of its sides 8; to find the other two sides.

2. Given R, r, and the distance of the inscribed centre from the vertex, to find the base.

3. Given R, r, and the perpendicular from the vertex to the base, to find the base.

4. Given R, r, and ρ, to find the base.

5. Given R, r, and the perimeter, to find the corresponding triangle.

6. Given R, ρ, and the excess of the other two sides above the base, to find the corresponding triangle.

7. Given R and r, to find the triangle when its area is a maximum or a minimum.

8. Given the vertical angle, r, and ρ, to find the base.

9. Given the vertical angle, the sum of the sides containing it, and the perpendicular from it to the base, to find the base.

10. Given the vertical angle, the perpendicular from it to the base, and the perimeter, to find the base.

11. Given the vertical angle, the difference of the sides containing it, and the perimeter, to find the base.

12. Given the vertical angle, the product of the sides containing it, and the perimeter, to find the base.

13. Given the vertical angle, the sum, and the product of the three sides, to find the base.

14. Given the vertical angle, the distance of the inscribed centre from the vertex, and d, to find the base.

15. Given the vertical angle, d, and δ, to find the base.

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16. Given the vertical angle and d, to find the triangle when the base is the greatest possible.

17. Given r, ρ, and the sum of the two sides, to find the base.

18. Given r, ρ, and the difference of the two sides, to find the base.

19. Given the three escribed radii, to find the triangle.

20. Given the inscribed radius and two escribed radii, to find the triangle.

Leonhard Euler (1765).

In the *Novi Commentarii Academiae Scientiarum Imperialis Petropolitanae*, Tom. xi. (pp. 103–123), for the year 1765, occurs a paper* of Euler's, entitled, *Solutio facilis problematum quorumdam geometricorum difficillimorum.* He there determines the distance between the inscribed and circumscribed centres of a triangle in the following way.

See figure 73.

Let I and S be the inscribed and circumscribed centres of the triangle ABC. Draw IF, SL perpendicular to AB; join AS, and draw AM perpendicular to BC.

Denote BC, CA, AB by a, b, c, and let area $= A$, $a+b+c=p$, $ab+ac+bc=q$, $abc=r$.

Then $$IF=\frac{2A}{a+b+c}, \quad AF=\frac{c+b-a}{2}.$$

From the similar triangles ACM, ASL,

$$AM : CM = AL : SL;$$

that is, $$\frac{2A}{a} : \frac{a^2+b^2-c^2}{2a} = \frac{c}{2} : SL;$$

therefore $$SL=\frac{c(a^2+b^2-c^2)}{8A}.$$

Hence $$AF-AL=\frac{c+b-a}{2}-\frac{c}{2}=\frac{b-a}{2};$$

and $$IF-SL=\frac{2A}{a+b+c}-\frac{c(a^2+b^2-c^2)}{8A},$$

$$=\frac{(a+b)c^3+(a^2+b^2)c^2-(a+b)(a^2+b^2)c-(a^2-b^2)^2}{8(a+b+c)A}.$$

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*An abstract is given of it in the *Proceedings* for last year. See Vol. IV., pp. 51-55.

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$$\text{Now IS} = (\text{Ar} - \text{AL})^2 + (\text{IF} - \text{SL})^2,$$

$$= \frac{abc}{16(a+b+c)^2\text{A}^2} \left\{ \begin{matrix} +a^5+a^4b+ab^4+abc^3-2a^3b^2-2a^2b^3 \\ +b^5+a^4c+ac^4+ab^3c-2a^3c^2-2a^2c^3 \\ +c^5+b^4c+bc^4+a^3bc-2b^3c-2^2b^2c^3 \end{matrix} \right\},$$

$$= \frac{abc}{16(a+b+c)\text{A}^2} \left\{ \begin{matrix} +a^4+a^2bc+2a^2b^2 \\ +b^4+ab^2c-2a^2c^2 \\ +c^4+abc^2-2b^2c^2 \end{matrix} \right\},$$

$$= \frac{r}{16p\text{A}^2}(p^4 - 4p^2q + 9pr),$$

$$= \frac{r(p^3 - 4pq + 9r)}{16\text{A}^2},$$

$$= \frac{r^2}{16\text{A}^2} - \frac{r}{p}.$$

Nicolas Fuss (1794, 1798).

The inquiry is now extended to other figures than the triangle. In the tenth volume, pp. 103–125, of the *Nova Acta Academiae Scientiarum Imperialis Petropolitanae* (Petropoli, 1797), there is a paper by Fuss, entitled, *De Quadrilateris quibus circulum tam inscribere quam circumscribere licet*, which was read on the 14th August 1794. The following is a short abstract of it. The words "encyclic" and "pericyclic" are used for the phrases "that may be inscribed in a circle," and "that may be circumscribed about a circle."

The first problem and its corollaries show how, when the four sides of an encyclic quadrilateral are given, to express by means of the sides alone, the area, the radius of the circumscribed circle, the diagonals, and the angles of the figure. When, however, the quadrilateral is pericyclic, in addition to the four sides there must be given either a point of contact or an angle to determine the rest. This is shown in the second and third problems and their corollaries. In the fourth problem and its corollaries it is proved that, of all the quadrilaterals formed by four given sides, the one whose inscribed circle is the greatest is encyclic. The fifth problem is, About a given circle to circumscribe an encyclic quadrilateral; the sixth, In a given circle to inscribe a pericyclic quadrilateral. The seventh and eighth problems show how, when the four sides, or when two sides and the contained angle, are given to construct a quadrilateral

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which is both encyclic and pericyclic. The ninth and tenth problems show how, when the angles of an encyclic and pericyclic quadrilateral are given, to find the sides, the area, and the radii of the circumscribed and inscribed circles. The eleventh problem is, Given the inscribed and circumscribed radii of an encyclic and pericyclic quadrilateral, to find the distance of the centres. Two expressions are found for the square of this distance, namely,

$$R^2 + r^2 \pm r\sqrt{4^2 + rR^2},$$

where the upper sign holds for the circle which touches the sides produced of the quadrilateral, and the lower for the circle really inscribed in the quadrilateral. In the latter case Fuss remarks that $R^2 + r^2$ must be greater than $r\sqrt{4R^2 + r^2}$, that is, that R must be greater than $r\sqrt{2}$. He then adds the expression for the square of the distance between the inscribed and circumscribed centres of a triangle, namely, $R^2 - 2Rr$,
and gives the following demonstration.

See figure 72.

Let I be the centre of the circle inscribed in triangle ABC, S the circumscribed centre. From A draw AV through the centre I to meet the circumscribed circle in V. Through V draw the circumscribed diameter VU, and from I let fall IT perpendicular to VU.

Then $d^2 = SI^2 = ST^2 + IT^2$.
But $ST = SV - TV = R - TV$,
and $IT^2 = IV^2 - TV^2$;
therefore $d^2 = R^2 - 2R \cdot TV + IV^2$.

Draw the chords BU, BV, CV.
Then it is evident that $\angle CHV = \angle UBV$,
and $\angle HCV = \angle BUV$;
therefore triangle BUV is similar to triangle HCV;
therefore $BV : UV = HV : CV$;
therefore $BV \cdot CV = UV \cdot HV$,
$= 2R \cdot HV$.

Now if CI be joined, it is clear that
$\angle CIV = \angle IAC + \angle ICA$,
and $\angle ICV = \angle ICH + \angle VCH$.
But $\angle ICH = \angle ICA$, and $\angle VCH = \angle IAC$;
therefore $\angle CIV = \angle ICV$,

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and $IV = CV = BV.$

Hence $IV^2 = BV \cdot CV,$

$= 2R \cdot HV.$

Substituting this value, we have

$$d^2 = R^2 - 2R \cdot (TV - HV),$$
$$= R^2 - 2R \cdot ID,$$
$$= R^2 - 2Rr.$$

In the thirteenth volume, pp. 166–189, of the *Nova Acta* (Petropoli, 1802), Fuss returns to the inquiry regarding the inscription and circumscription of circles to certain polygons. The title of his paper is *De Polygonis symmetrice irregularibus circulo simul inscriptis et circumscriptis;* it was read on the 19th April 1798. He says that he tried various ways of resolving the same problems, as he had done in the case of the quadrilateral, for polygons of more than four sides, but without success. The fundamental formulae became so perplexed that oil and toil ("oleum et opera") were fruitlessly spent in clearing them up. Laying aside, therefore, the general problem as beset with very great difficulties, he betook himself to polygons, which may be called *symetrically irregular*, that is, which have a diameter passing through the centres of the inscribed and circumscribed circles, and dividing the polygons into two equal and similar figures. Even with this limitation, he observes, the problem is a knotty one.

The following are the enunciations of the ten problems of which the paper consists:—

1. Given the angles of a pentagon, to find its sides so that a circle of given size can be circumscribed about it.

2. Given the angles of a pentagon, to find its sides so that a circle of given size can be inscribed in it.

3. Given the sides of a pentagon and any one point of contact, to find the radius of the inscribed circle.

4. If the angles of an encyclic and pericyclic pentagon are known, to find the relation between the radii R and r.

5. To find the distance of the centres of the circles inscribed and circumscribed to a symmetrically irregular pentagon.

6. In a given circle to inscribe a symmetrically irregular pentagon which shall be pericyclic.

7. To find the distance of the centres of the circles inscribed and circumscribed to a symmetrically irregular hexagon.

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8. In a given circle to inscribe a symmetrically irregular hexagon which shall be pericyclic.

9. In a given circle to inscribe a symmetrically irregular heptagon which shall be pericyclic.

10. In a given circle to inscribe a symmetrically irregular octagon which shall be pericyclic.

Geometrical Notes.

By R. E. Allardice, M.A.

1. Some points of difference between polygons with an even number and polygons with an odd number of sides.

A polygon with an odd number of sides is determined when its angles are given and it is such that a circle of given radius may be circumscribed about it; while a polygon with an even number of sides is not determined by these conditions.

To prove this it is sufficient to show that in general one polygon and only one with given angles can be inscribed in a given circle if the number of sides be odd; and that if the number of sides be even either it is impossible to inscribe any such polygon or it is possible to inscribe an infinite number.

Let ABCDE (fig. 76) be a polygon with an odd number of sides. In a circle take any point A'; make an arc A'C' to contain an angle equal to B; an arc C'E' to contain an angle equal to D; an arc E'B' to contain an angle equal to A; and so on. Thus the problem is in general a possible one; but it is evident that, though no definite relation among the angles is required (except that connecting the angles of any n-gon), there are limits in which the angles must lie in order that a solution be possible. Thus when the above construction is made, the point E' must fall between C' and A', the point B', between A' and C', and so on.

If the angles are $A_1, A_2, A_3 \ldots\ldots A_{2n+1}$, the necessary and sufficient conditions are of the form

$$A_2 + A_4 + \ldots\ldots + A_{2n} > (n-1)\pi$$
$$A_2 + A + \ldots\ldots + A_{2n} + A_1 < n\pi.$$

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Consider next a polygon ABCDEF (Fig. 77) with an even number of sides. Take any point A′ in a circle; make the arc A′C′ to contain an angle equal to B; an arc C′E′ to contain an angle equal to D; then if the arc E′A′ contains an angle equal to F, but not otherwise, the problem is possible. Suppose this condition to be satisfied. Take any point B′ in the arc A′C′; make the arc B′D′ to contain an angle equal to C; and the arc D′F′ to contain an angle equal to E. Then A′B′C′D′E′F′ and ABCDEF are equiangular. Hence either there is no solution or there are an infinite number.

The above condition is equivalent to $B+D+F=2\pi$, or, in the case of a polygon with $2n$ sides,

$$A_2+A_4+\ldots\ldots+A_{2n}=(n-1)\pi.$$

It may be shown that the above data involve, in both cases, the number of conditions necessary to determine an n-gon. For $n-3$ conditions must be satisfied in order that a circle may be drawn to circumscribe a given n-gon; and one more, in all $n-2$ conditions, in order that a *given* circle may be so drawn. These with the $n-1$ conditions involved in the giving of the angles, give the $2n-3$ necessary conditions. In the case of a polygon with an even number of sides, one of the conditions that a circle may be drawn to circumscribe it, is expressible in terms of the angles of the polygon alone; while for a polygon with an odd number of sides, this is not the case.

A particular case of the above is that an equi-angular polygon inscribed in a circle is necessarily regular if it have an odd number of sides, but not if it have an even number of sides.

This particular case is given in Charles Hutton's *Miscellanea Mathematica*, London, 1775, p. 271, in the form:—

"An equilateral figure inscribed in a circle is always equiangular.

"But an equiangular figure inscribed in a circle is not always equilateral, except when the number of sides is odd. If the sides be of an even number, then they may be either all equal, or else half of them will always be equal to each other, the equals being placed alternately."

[I am indebted for this reference to Dr Mackay.]

The dual of this theorem, modified slightly, as sides instead of angles are now concerned, is also true, namely—

It is, in general, possible to describe one and only one n-gon with n given lines for sides such that a circle may be inscribed in it, if n

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be an odd number; but if n be an even number, either there is no such n-gon or there are an infinite number.

The case of a pentagon may be selected as the method of proof is general.

Let AB, BC, &c. (fig. 78), be the given sides; then points G, H, K, L, M, may be found in one way and in one way only, such that AG = AM, BH = BG, &c.

Let $\quad AG = x_1$, $BH = x_2$, &c.
$\quad AB = a$, $\quad BC = b$, &c.

Then $\quad x_1 + x_2 = a \qquad x_2 + x_3 = b \qquad x_3 + x_4 = c$
$\quad x_4 + x_5 = d \qquad x_5 + x_1 = d$;

which equations give unique values for x_1, x_2, &c., namely,

$$2x_1 = a - b + c - d + e, \text{ &c.}$$

Suppose, now, that a pentagon could be found with these sides such that a circle could be inscribed in it (fig. 78).

Let $AOG = \alpha$, $BOH = \beta$, &c.

It may be shown that α, β, &c., and the radius of the circle may be determined uniquely, as follows:—

It is evident that $\alpha + \beta + \gamma + \delta + \epsilon = \pi$.

Take any line (fig. 79), and from a point P in it cut off parts PQ_1, PQ_2, &c., equal to x_1, x_2, &c.

Draw a perpendicular through P to this line; then we have to find a point R in this line, such that

$$PRQ_1 + PRQ_2 + \ldots\ldots = \pi.$$

It is evident that one such position and only one exists, since when R is at P each of these angles is a right angle, and when R is at an infinite distance each of them is zero, and each of the angles diminishes continuously as R moves away from P.

It is obvious that PR is the radius of the inscribed circle and that PRQ_1, PRQ_2, &c., are the angles subtended at the centre by x_1, x_2, &c.

There is thus one and only one solution of the problem.

If, however, there is an even number of sides (say six) there are the equations

$$x_1 + x_2 = a \qquad x_2 + x_3 = b \qquad x_3 + x_4 = c$$
$$x_4 + x_5 = d \qquad x_5 + x_6 = e \qquad x_6 + x_1 = f$$

which give $\quad a - b + c - d + e - f = 0$;

so that, unless this relation is satisfied, it is impossible to find any

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polygon of the kind required; and if it is satisfied, there are an infinite number.

For in AB (fig. 80) take any point G; make BH = BG, CK = CH, ...FN = FM; then, on account of the above relation, AG will be equal to AN.

There are thus an infinite number of ways of dividing the sides as required; and, corresponding to each method of division, it is possible to construct a hexagon in which a circle may be inscribed, by finding the radius of the inscribed circle and the angles subtended at the centre by the segments of the sides, as in the case of a polygon of an odd number of sides.

Corollary. An equilateral polygon circumscribing a circle is necessarily regular, if it have an odd number of sides; but if it have an even number, this is not necessarily the case. In the latter case, however, alternate angles are equal.

It is obvious that an equiangular polygon circumscribing a circle is necessarily regular.

Some other cases of differences between polygons with an even and those with an odd number of sides are well known. Thus if all the sides of a polygon of the first kind are produced, a crossed polygon is formed; while if the sides of one of the second kind are produced, two polygons are formed.

There are also differences in the formation of the different crossed n-gons with the sides of a given n-gon, according as n is a prime number or not.

Again, when the middle points of the sides of a polygon are given, one and only one polygon can in general be formed, if the number of sides is odd; but either none or an infinite number, if the number of sides is even. [Dr Mackay gives me a reference for this theorem to Catalan's *Théorèmes et Problèmes de Géométrie Elémentaire* (6th ed., pp. 10, 11); and Catalan gives a reference to Prouhet, *Nouvelles Annales*, tome III. The latter reference I have verified.]

The treatment of this latter question by means of co-ordinates is perhaps of some interest.

Let (x_1, y_1), (x_2, y_2), &c, be the co-ordinates of the vertices required; (a_1, b_1), (a_2, b_2), &c., those of the middle points.
Then

$$x_1 + x_2 = 2a_1$$
$$x_2 + x_3 = 2a_2, \text{ &c.}$$

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These equations have a determinate solution if n is odd, namely,

$$x_1 = a_1 - a_2 + a_3 - a_4 + \ldots\ldots, \text{ \&c.};$$

but if n is even there is no solution unless

$$a_1 - a_2 + a_3 - a_4 + \ldots\ldots = 0$$

or

$$a_1 + a_3 + \ldots\ldots = a_2 + a_4 + \ldots\ldots,$$

which means that the centres of gravity of the two polygons, got by taking the given points alternately, coincide. If this condition is satisfied, there are an infinite number of solutions. It may be noted that the given points may be taken in any order, with the exception that when n is even, the points taken once in odd order must always be taken in odd order.

II. If a straight line (fig. 81) meets the three diagonals AC, BD, EF, of a complete quadrilateral in the points L, M, N, and if L′, M′, N′, are the harmonic conjugates of L, M, N, with respect to A and C, B and D, E and F respectively, then L′, M′, N′, are collinear.

Since P and Q, and also M and M′, are harmonically conjugate with respect to B and D, DM′PBQM is an involution of which B and D are the double points.

Hence $$(BMDQ) = (BM'DP),$$

that is, $$\frac{BD.MQ}{BQ.MD} = \frac{BD.M'P}{BP.M'D};$$

$\therefore$ $$\frac{M'P}{MQ} = \frac{BP}{BQ}.\frac{M'D}{MD}.$$

Similarly from the relation

$$(BM'DQ) = (BMDP),$$

$$\frac{MP}{M'Q} = \frac{BP}{BQ}.\frac{MD}{M'D};$$

$\therefore$ $$\frac{MP}{MQ} = \frac{M'Q}{M'P}.\frac{BP^2}{BQ^2}.$$

$\therefore$ $$\frac{MP}{MQ}.\frac{LR}{LP}.\frac{NQ}{NR} = \frac{M'Q}{M'P}.\frac{L'P}{L'R}.\frac{N'R}{N'Q}.\left(\frac{BP}{BQ}.\frac{CR}{CP}.\frac{EQ}{ER}\right)^2$$

Now $$\frac{MP}{MQ}.\frac{LR}{LP}.\frac{NQ}{NR} = 1, \text{ and } \frac{BP}{BQ}.\frac{CR}{CP}.\frac{EQ}{ER} = 1;$$

$\therefore$ $$\frac{M'Q}{M'P}.\frac{L'P}{L'R}.\frac{N'R}{N'Q} = 1;$$

and hence, L′, M′, N′, are collinear.

Cor. 1. The middle points of the diagonals are collinear.

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Cor. 2. If the first corollary be proved independently, the theorem may be deduced from it by projection.

Cor. 3. The dual of this theorem is also true. It may be stated as follows:—If lines l, m, n, be drawn through any point to the three diagonal points of a complete quadrangle, and the harmonic conjugates l', m', n', of l, m, n, be taken with respect to the sides of the quadrangle that pass through these points, then l', m', n', will be concurrent.

Sixth Meeting, April 8th, 1887.

J. S. MACKAY, Esq., LL.D., in the Chair.

On the value of $\Delta^n 0^m / n^m$, when m and n are very large.

By Professor TAIT.

I had occasion, lately, to consider the following question connected with the Kinetic Theory of Gases:—

Given that there are 3.10^{20} particles in a cubic inch of air, and that each has on the average 10^{10} collisions per second; after what period of time is it even betting that any specified particle shall have collided, once at least, with each of the others?

The question obviously reduces to this:—Find m so that the terms in

$$X^m = (x_1 + x_2 + x_3 + \ldots\ldots + x_n)^m$$

which contain each of the n quantities, once at least, as a factor, shall be numerically equal to half the whole value of the expression when $x_1 = x_2 = \ldots\ldots = x_n = 1$. Thus we have

$$X^m - \Sigma(X - x_r)^m + \Sigma(X - x_r - x_s)^m - \ldots\ldots = \tfrac{1}{2}X^m$$

or

$$\Delta^n 0^m / n^m = \tfrac{1}{2}.$$

It is strange that neither Herschel, De Morgan, nor Boole, while treating differences of zero, has thought fit to state that Laplace had, long ago, given all that is necessary for the solution of such questions. The numbers $\Delta^n 0^m$ are of such importance that one would naturally expect to find in any treatise which refers to them at least a state-

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ment that in the *Théorie Analytique des Probabilités* (Livre II., chap. ii., § 4) a closely approximate formula is given for their easy calculation. No doubt the process by which this formula is obtained is somewhat difficult as well as troublesome, but the existence of the formula itself should be generally known.

When it is applied to the above problem, it gives the answer in the somewhat startling form of "about 40,000 years."

P.S.—April 4, 1887.—Finding that Laplace's formula ceases to give approximate results, for very large values of m and n when these numbers are of the same order of magnitude, I applied to Prof. Cayley on the subject. He has supplied the requisite modification of the formula, and his paper has been to-night communicated to the *Royal Society of Edinburgh*.

Sur les cordes communes à une conique et à un cercle de rayon nul; Application à la théorie géométrique des foyers dans les coniques.

Par M. Maurice d'Ocagne.

1. Etant donnés une conique K, dont l'équation est $K = 0$, et un point P (α, β), l'équation générale des coniques qui passent par les points d'intersection de la conique K et du cercle P de rayon nul, qui a le point (α, β) pour centre, est

$$K + \lambda[(x-\alpha)^2 + (y-\beta)^2] = 0. \tag{1}$$

Comme les quatre points d'intersection du cercle P et de la conique K sont imaginaires, le système (1) comprend un seul couple de droites réelles Δ et Δ'. Ces droites seront dites, par analogie avec une expression proposée par Chasles,* les *conjointes du point P et de la conique K.*

Parmi les couples de cordes imaginaires communes au cercle P et à la conique K se trouvent les *droites isotropes* passant au point P

$$(x-\alpha) + i(y-\beta) = 0$$
$$(x-\alpha) - i(y-\beta) = 0,$$

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* *Journal de Liouville*, T. III., p. 385.

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c'est-a-dire, les droites qui unissent le point P aux *ombilics* I et J du plan (expression de Laguerre), points imaginaires situés sur la droite de l'infini et par où passent tous les cercles du plan.

Une première conséquence de cette remarque est que *le point P a même polaire relativement à la conique K et aux conjointes* Δ *et* Δ'. Les conjointes passent en effet, d'après ce qui précède, par les points communs à la conique K et aux droites PI et PJ.

Par conséquent, le point de rencontre des conjointes Δ et Δ' se trouve sur la polaire du point P relativement à la conique K. On peut observer, aussi, qu'en vertu d'une propriété générale des coniques passant par l'intersection d'une conique et d'un cercle, *les conjointes* Δ *et* Δ' *sont également inclinées sur les axes de la conique K.*

Une deuxième conséquence de la remarque faite plus haut est que parmi les conjointes d'un point et d'un cercle se trouve toujours la droite à l'infini du plan. L'autre conjointe, située à distance finie, est *l'axe radical du point et du cercle*, dont les propriétés sont bien connues.

On voit tout de suite que les conjointes du centre d'une ellipse

$$x^2/a^2 + y^2/b^2 = 1 \qquad (a > b)$$

et de cette courbe sont données par

$$y = \pm ab/c\,;$$

et celles du centre d'une hyperbole

$$x^2/a^2 - y^2/b^2 = 1,$$

par

$$x = \pm ab/c.$$

Dans le cas de l'ellipse, *les tangentes menées des extrémités du grand axe au cercle qui a pour diamètre le petit axe, coupent ce petit axe aux points par où passent les conjointes du centre, qui sont, d'ailleurs, parallèles au grand axe.* C'est la traduction de la formule

$$y = \pm ab/c.$$

2. L'importance de la considération des conjointes réside dans le théorème suivant :—

Dans la transformation par polaires réciproques relativement à un cercle, les éléments corrélatifs des foyers d'une conique sont les conjointes du centre du cercle directeur (centre de la transformation) et de la conique corrélative.

La démonstration de ce théorème est des plus simples. Les ombilics I et J étant situés sur la droite de l'infini, et les directions

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isotropes OI et OJ (O est le centre de la transformation) étant rectangulaires, les ombilics I et J ont respectivement pour éléments corrélatifs les droites OJ et OI. Or, les foyers d'une conique K sont les points de rencontre réels des couples de tangentes menées à K par les ombilics I et J. Les éléments corrélatifs de ces foyers seront donc les cordes communes à la conique corrélative de K et au couple de droites OJ et OI, c'est-à-dire, les conjointes du centre O de la transformation, et de la conique corrélative de K.

A titre de corollaire immédiat de ce théorème on peut remarquer que si l'un des foyers de la conique K coïncide avec le centre O de la transformation, l'une des conjointes du point O et de la conique corrélative étant rejetée à l'infini, cette conique corrélative est un cercle, résultat bien connu dont on pénètre ainsi la raison intime.

Le théorème précédent permettra de transformer les propriétés des foyers des coniques en propriétés de conjointes, et *vice versâ*. En particulier, on pourra considérer les systèmes de coniques ayant les mêmes cordes réelles communes avec un point donné, coniques qui pourront être dites *homoconjonctives* par rapport à ce point, et toutes leurs propriétés se déduiront corrélativement des propriétés bien connues des systèmes de coniques homofocales.

Mais ici nous nous attacherons surtout—c'est là le principal objet de ce petit Mémoire—à faire voir comment le théorème précédent peut être utilisé pour déduire les propriétés focales des coniques des propriétés tout élémentaires de l'axe radical d'un cercle et d'un point.

3. Soient C un cercle, O un point, f l'axe radical de ce cercle et de ce point (droite équidistante du point O et de la polaire de ce point relativement au cercle C).* Une transformation par polaires réciproques de centre O donne comme courbe corrélative du cercle C une conique K, qui est une ellipse, une hyperbole ou une parabole, selon que le point O est à l'intérieur, à l'extérieur ou sur la circonférence du cercle C. Mais, dans tous les cas, cette conique K a pour foyers le point O et le point F corrélatif de la droite f.

4. Joignons le point O à un point M pris sur le cercle C; la tangente en M au cercle C coupe la droite f au point N; tirons ON et menons $O\mu$ parallèlement à MN.

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* Le lecteur est prié de faire les figures.

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D'après une propriété fondamentale de l'axe radical f du point O et du cercle C, on a $NO = NM$;

donc $\angle NOM = \angle NMO$, et $\angle NOM = \angle MO\mu$.*

L'élément corrélatif du point M pris sur le cercle C est une tangente m à la conique K ; celui de la droite f est le foyer F de la conique K, l'autre foyer étant au point O ; celui de la tangente MN au cercle C est le point de contact P de la tangente m sur la conique K ; ceux des points N et μ (ce dernier situé à l'infini dans la direction MN) sont les rayons vecteurs PF et PO. La transformation de la propriété précédente montre donc que l'angle de PF avec la tangente m est égal à l'angle de m avec PO. On obtient ainsi cette propriété classique :—

La tangente en un point d'une conique est bissectrice de l'angle des rayons vecteurs qui unissent le point de contact aux foyers de la conique.

5. Les polaires d'un point M, pris sur l'axe radical de deux cercles, relativement à ces deux cercles, se coupent sur leur axe radical, car elles sont elles-mêmes les axes radicaux du cercle de centre M orthogonal aux cercles donnés et de ces cercles. Lorsque l'un des cercles donnés se réduit à un point O cette propriété devient : si la polaire, relativement à un cercle C d'un point M pris sur l'axe radical d'un point O et de ce cercle C coupe cet axe radical au point M', l'angle MOM' est droit. Transformant par polaires réciproques, on a ce théorème bien connu :—

Le pôle d'une droite passant par le foyer F d'une conique, relativement à cette conique est sur la perpendiculaire élevée en F à cette droite.

Ou bien :—

La perpendiculaire menée par un foyer F d'une conique au rayon vecteur d'un point M de cette conique coupe la tangente en M sur la directrice relative au foyer F.†

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* Une transformation homographique permet de déduire de là le théorème suivant :—*Les segments d'une tangente à une conique, compris entre le point de contact de cette tangente et ses intersections avec les conjointes d'un point par rapport à la conique, sont vus de ce point sous des angles égaux.*

† On peut aussi remarquer, en observant que l'élément corrélatif du centre d'un cercle C est la directrice de la conique corrélative K qui correspond au foyer confondu avec le centre O de la transformation, que cette propriété est également corrélative de celle-ci :—

Toutes les normales à un cercle passent par le centre de ce cercle.

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6. Tout cercle dont le centre M est sur la droite f et qui passe par le point O coupe orthogonalement le cercle C, c'est-à-dire qu'il passe par les points de contact des tangentes menées de M au cercle C. La transformation par polaires réciproques de cette propriété montre que si F et F' sont les foyers d'une conique K et que MM' soit une corde de la conique passant par le foyer F, la parabole qui a F' pour foyer et MM' pour directrice est tangente aux tangentes menées à la conique K par les points M et M'. En outre, si μ et μ' sont les points de contact de ces tangentes et de la parabole les angles MF'μ et M'F'μ' sont droits. Donc, d'après le théorème qui termine le No. 5, les points μ et μ' appartiennent à la directrice de la conique K relative au foyer F'. On est ainsi conduit à ce théorème qui ne nous semble pas avoir été déjà remarqué.

Une parabole qui a pour foyer un foyer F' d'une conique K et pour directrice une droite quelconque passant par l'autre foyer F de la conique K et coupant cette conique aux points M et M', est tangente aux tangentes à la conique K menées par les points M et M', la corde de contact étant la directrice de la conique K relative au foyer F'.

7. Soient f et f_1 les axes radicaux d'un point O et de deux cercles C et C_1. La perpendiculaire abaissée du point de rencontre de f et f_1 sur la ligne des centres de C et C_1 est l'axe radical de ces deux cercles. La transformation par polaires réciproques, le point O étant toujours pris pour centre de la transformation, donne ce théorème qui nous semble également nouveau :—

Si deux coniques ont en commun un foyer O, ces coniques ont un pôle double réel qui est à la rencontre de la droite qui joint les autres foyers F et F_1 et de la perpendiculaire élevée en O à la droite qui joint ce foyer au point de rencontre des directrices qui lui correspondent dans les deux coniques.

8. Si t et t' sont les tangentes menées d'un point M à un cercle C, et que D soit un point pris sur la polaire du point M relativement au cercle C, la polaire d du point D passe par le point M, et les droites d et MD sont conjuguées harmoniques par rapport aux droites t et t'.

Supposons alors que le point M se trouve sur l'axe radical f du point O et du cercle C, et que nous prenions pour point D le point de la polaire de M qui se trouve sur la polaire de O ; la droite d se confond alors avec MO. Soient T et T' les points de contact avec C

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des tangentes issues de M ; puisque MT = MT' = MO, si par le point O nous élevons à OT et à OT' les perpendiculaires ON et ON' qui coupent MT et MT' respectivement en N et en N', nous avons TN = 2TM, T'N' = 2T'M ; par suite, la droite NN' est symétrique de TT' par rapport à M, et si cette droite coupe MD en E, OE est parallèle à *f*.

Cela posé, opérons une transformation par polaires réciproques de centre O. Au point M situé sur *f* correspond une droite passant par le foyer F de la conique K et coupant cette conique en deux points A et A'. Aux points N et N' correspondent les normales à la conique K en A et A', normales qui se coupent en B. Au point E correspond la parallèle à l'axe focal de la conique K, menée par le point B ; si cette parallèle coupe AA' au point C, le point C est dès lors corrélatif de la droite ME ; mais nous venons de voir que la droite MD' est conjuguée harmonique de MO par rapport aux tangentes MT et MT' ; donc le point C est conjugué harmonique du point situé à l'infini sur AA' par rapport aux points A et A', c'est-à-dire que le point M est le milieu de AA', et nous obtenons ce théorème connu :—

Si par le point de rencontre des normales à une conique menées par les extrémités d'une corde focale, on mène une parallèle à l'axe focal de cette conique, cette droite passe par le milieu de la corde focale considérée.

9. Supposant toujours le point M situé sur l'axe radical *f* du point O et du cercle C, plaçons, maintenant, le point D à la rencontre de la polaire du point M relativement au cercle C et de la droite qui joint le centre de ce cercle au point O—c'est-a-dire, au pôle de la droite *f*. La droite *d* coincide alors avec la droite *f*, et nous voyons que MD est conjuguée harmonique de *f* par rapport aux tangentes au cercle C issues de M, ou, si ces tangentes coupent en H et en H' la parallèle à *f* menée par D, que D est le milieu de HH', et, par suite, que les angles HOD et DOH' sont égaux.

Transformons toujours par polaires réciproques :

Au pôle D de la droite *f* correspond la directrice de la conique K relative au foyer F. On a donc ce théorème connu :

Si la corde AA' d'une conique passe par le foyer F de cette conique, les droites, qui joignent les extrémités A et A' de cette corde au point de rencontre de la directrice relative au foyer F et de l'axe focal, sont également inclinées sur cet axe.

10. Considerons un cercle C de centre Ω (fig. 82), un point O et

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une droite d quelconque perpendiculaire à $O\Omega$. Joignons le point O à un point P mobile sur le cercle C; menons OQ perpendiculaire à OP et QR parallèle à la tangente PT menée en P au cercle C, c'est-à-dire perpendiculaire à $P\Omega$. Nous avons

$$\angle OQR = \angle OP\Omega = \theta, \quad \angle OQH = \angle PO\Omega = \phi.$$

Donc, si nous abaissons du point O sur QR la perpendiculaire OR, nous avons

$$OR = OQ\sin\theta = OH\frac{\sin\theta}{\sin\phi} = OH\frac{O\Omega}{P\Omega},$$

et comme OH, $O\Omega$ et $P\Omega$ sont constants, OR est aussi constant; par suite, la droite QR enveloppe un cercle de centre O.

Transformons par polaires réciproques. Nous avons une conique K ayant un foyer au point O. Au point P correspond une tangente t a cette conique; à la droite d, un point Δ de l'axe focal; au point Q, la perpendiculaire p abaissée de Δ sur t; à la droite QR qui joint le point Q au point situé à l'infini sur la tangente PT au cercle C, le point de rencontre de la droite p et du vecteur qui joint le foyer O au point de contact de la tangente t et de la conique K. D'ailleurs le cercle de centre O enveloppé par QR a pour corrélatif également un cercle de centre O. On a donc ce théorème connu :—

Le lieu du point de rencontre du rayon vecteur qui unit un point M mobile sur une conique à l'un des foyers de cette conique et de la perpendiculaire abaissée d'un point Δ de l'axe focal sur la tangente au point M, est un cercle ayant pour centre le foyer O.

Si on prend pour point Δ le second foyer F de la conique, on voit, en considérant le point M dans une position infiniment voisine de l'un des sommets de l'axe focal, que le rayon du cercle correspondant (cercle directeur) est égal à la longueur de cet axe. Rapprochant ce résultat du théorème obtenu au No. 4, on en déduit la propriété des rayons vecteurs dans les coniques.

11. Nous donnerons encore un exemple remarquable de la méthode que nous indiquons ici.

Supposons que le cercle C soit variable, mais ait constamment avec le point O même axe radical f. Considérons en outre un point fixe quelconque D. L'axe radical des points D et O considérés comme cercles de rayon nul est la perpendiculaire élevée à DO en son milieu. Cette droite coupe f en un point K, et l'axe radical du point D et du cercle C passe constamment par le point K.

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Il en résulte que la polaire du point D relativement au cercle C passe constamment par le point D′ symétrique du point D par rapport au point K, point situé à la rencontre de DK et de la perpendiculaire élevée en O à OD.

Passons à la figure corrélative. Aux différents cercles C correspondent des coniques ayant toutes un foyer au point O et un foyer au point F, c'est-a-dire des coniques *homofocales*, et si d est la droite corrélative du point D, nous voyons que les pôles de cette droite par rapport aux coniques du système sont situés sur une droite d', la corrélative du point D′. Le point K a pour élément corrélatif la droite qui joint le point F au symétrique du point O par rapport à la droite d. Cette droite coupe la droite d en un point M, et puisque l'angle DOD′ est droit, la droite d est la perpendiculaire élevée en M à la droite d. De là ce théorème connu :—

Les pôles d'une droite d relativement à un système de coniques homofocales sont situés sur la perpendiculaire menée à cette droite par le point où elle est coupée par la droite qui joint l'un des foyers au symétrique de l'autre par rapport à d.

Remarquant que la droite d est tangente au point M à une conique ayant pour foyers F et O, on peut énoncer encore ce théorème de la manière que voici.

Le lieu des pôles d'une droite d relativement à un système de coniques homofocales est la normale à celle de ces coniques qui touche la droite d, menée par le point de contact de cette conique et de cette droite.

12. Nous nous bornerons aux exemples qui précèdent pour mettre en relief la fécondité de la méthode qui consiste à déduire les propriétés focales des coniques de la théorie des axes radicaux.

Pour terminer, nous ferons observer que réciproquement toute propriété des foyers conduit corrélativement à une propriété des conjointes d'un point et d'une conique, et, plus particulièrement, de l'axe radical d'un point et d'un cercle.

Exemple.—Prenons cette propriété connue : Soit MFM′ une corde focale de l'ellipse dont le grand axe est AA′ ; si l'on prolonge MA et M′A jusqu'à leurs points de rencontre Q et Q avec la directrice qui correspond au foyer F, l'angle QFQ′ est droit.*

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* Rouché et de Comberousse, *Traité de Géométrie*, T. II., 5e édit., p. 529, Ex. 916.

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Appelons O le second foyer de la conique, et transformons par polaires réciproques en prenant le point O pour centre de la transformation. Nous obtenons ainsi ce théorème :—

Soient f l'axe radical d'un point O et d'un cercle C, et P le pôle de cet axe relativement au cercle C. Si les tangentes menées d'un point quelconque de f au cercle C coupent l'une des tangentes à ce cercle parallèles à f aux points I et I', et que les droites PI et PI' coupent la droite f aux points H et H', l'angle HOH' est droit.

Note on the Kinematics of a Quadrilateral.

By R. J. Dallas.

I send a note on the following problem, a solution of which was requested of me by one of the tutors at King's College, Cambridge.

We are given a quadrilateral of four jointed bars ABCD (fig. 83). The bar CD being held fast, find the tangent to the locus of P, the intersection of DA, CB in any position ; and verify the following construction for the radius of curvature of the path of P :—

Let PQ be the third diagonal, draw through P a perpendicular to PQ meeting BA, CD in L and L' ; through L and L' draw parallels to PQ meeting AD in M and M' ; through M and M' draw perpendiculars to AD meeting the normal at P in O and O' ; then will

$$-1/\rho = 1/OP + 2/O'P.$$

The first part of this is easily found. The important angles in the figure have been marked thus—

$$CPQ = \alpha,\ DPQ = \beta,\ AQP = \gamma,\ DQP = \epsilon.$$

Making PD rock through a small angle $\delta\theta$, we have, if δs is the resulting element of arc traced out by P,

$$\delta s = PD\delta\theta \operatorname{cosec} TPD, \text{ and so also, if } \phi = \angle PCD,$$

$$\delta s = PC\delta\phi \operatorname{cosec} TPC.$$

Now $\delta\phi/\delta\theta = QD/QC$ as is well known, (see Goodeve's *Elements of Mechanism* p. 110), and thus $\dfrac{\sin TPC}{\sin TPD} = \dfrac{QD}{PD}\dfrac{PC}{QC}$.

Now $$\frac{QD}{PD} = \frac{\sin DPQ}{\sin PQD} \text{ and } \frac{PC}{QC} = \frac{\sin PQC}{\sin QPC},$$

therefore $$\frac{\sin TPC}{\sin TPD} = \frac{\sin\beta}{\sin\alpha} \text{ and } TPC - TPD = \beta - \alpha ;$$

therefore $$TPC = \beta \text{ and } TPD = \alpha.$$

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Thus TP makes an angle with DP equal to the angle QPC.

The construction in the figure is equivalent to asserting that

$$1/\rho = \sin\alpha\sin\beta(2\cot\epsilon + \cot\gamma)/PQ.$$

Let PT make an angle ψ with CD, then $\psi = \pi - \theta - \alpha$ and thus $\delta\psi = -(\delta\theta + \delta\alpha)$.

Also $\delta s = PD\delta\theta\operatorname{cosec}\alpha$, sensibly. We therefore must find $\delta\alpha$.

We have $\phi = \epsilon + \alpha$; therefore $\delta\alpha = \delta\phi - d\epsilon$.

To find $\delta\epsilon$ we displace PQ twice, first the end P into its new position* and then the end Q. Let δDQ be the increment of DQ.

Then $\delta\epsilon = (\delta s\sin(\alpha+\beta) - \delta DQ\sin\epsilon)/QP$.

To find δDQ, fix AD instead of CD, and rock DQ through an angle $\delta\theta$, then the tangent to the path of Q makes as before an angle equal to PQA with DQ (as shown by dotted line) and $\delta DQ = DQ\delta\theta\cot\gamma$. Thus

$$\delta\epsilon = \frac{\delta\theta}{PQ}\left[\frac{PD\sin(\alpha+\beta)}{\sin\alpha} - \frac{DQ\sin\epsilon}{\tan\gamma}\right]$$

and thus $$\delta\psi = -(\delta\theta + \delta\alpha) = -(\delta\theta + \delta\phi - \delta\epsilon).$$

From this, dividing by ds, we get

$$\frac{1}{\rho} = \frac{\sin\alpha}{PD}\left[1 + \frac{DQ}{CQ} + \frac{DQ}{PQ} - \frac{DP}{PQ}\frac{\sin(\alpha+\beta)}{\sin\alpha}\right]$$

$$= \frac{\sin\alpha\sin\beta}{PQ}\left[\frac{PQ}{PD\sin\beta} + \frac{DQ.PQ}{CQ.PD\sin\beta} + \frac{DQ}{DP}\frac{\cot\gamma\sin\epsilon}{\sin\beta} - \frac{\sin(\alpha+\beta)}{\sin\alpha\sin\beta}\right]$$

$$= \frac{\sin\alpha\sin\beta}{PQ}\left[\frac{\sin\theta}{\sin\epsilon\sin\beta} + \frac{\sin\phi}{\sin\epsilon\sin\alpha} + \cot\gamma - \frac{\sin(\alpha+\beta)}{\sin\alpha\sin\beta}\right]$$

$$= \frac{\sin\alpha\sin\beta}{PQ} \cdot \frac{\sin\theta\sin\alpha\sin\gamma + \sin\phi\sin\beta\sin\gamma + \cos\gamma\sin\alpha\sin\beta\sin\epsilon - \sin(\alpha+\beta)\sin\gamma\sin\epsilon}{\sin\alpha\sin\beta\sin\gamma\sin\epsilon}$$

Substituting for θ and ϕ in terms of the other angles, namely, $\theta = \beta + \epsilon$, $\phi = \alpha + \epsilon$, we readily get the result

$$-1/\rho = \sin\alpha\sin\beta(2\cot\epsilon + \cot\gamma)/PQ,$$

which agrees with the result already given.

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* Evidently for this purpose we may treat the element of arc as straight.

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Seventh Meeting, May 13th, 1887.

GEORGE THOM, Esq., LL.D., President, in the Chair.

Some Correspondence between Robert Simson, Professor of Mathematics in the University of Glasgow, and Matthew Stewart, Professor of Mathematics in the University of Edinburgh.

BY JAMES TAYLOR, M.A.

This correspondence, purchased at the sale of the Gibson-Craig collection of MSS., is now in the possession of Dr J. S. Mackay.

An Experiment in the Teaching of Geometry.

BY A. Y. FRASER, M.A., F.R.S.E.

§ 1. The course of geometry here referred to was given to pupils in George Heriot's School in the session preceding that in which they should begin the usual systematic study of geometry. The chief object of the course was to furnish their minds with a number of geometrical ideas before they should meet with these ideas as treated by Euclid. Subsidiary ends were also kept in view—such as to get them to make neat and accurate figures, and to enable them to solve various practical problems of construction and measurement.

§ 2. The classes with whom the experiment was tried were three in number, containing each fifty boys. The average age of the boys at the middle of the session was 12·3, 13·2 and 13 years. The time given was one period of forty-five minutes per week.

§ 3. The school provided fifty pairs of compasses and fifty box-wood rulers ($9\frac{1}{2}''$ long, of special design, $\frac{1}{4}''$ at each end being unmarked and the inches being divided into 8ths, 10ths, 12ths, and 16ths). Each pupil had a note-book about the size of an ordinary copy book, ruled in squares $\frac{1}{3}'' \times \frac{1}{3}''$.

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§ 4. The method adopted in the lessons was in the main as follows:—First the problem was worked out on the blackboard, an attempt, which was usually successful, being always made to get the pupils to discover the solution for themselves. After the ground was cleared in this way, the pupils wrote down in their books a simple statement of the construction or measurement to be effected, and this they carried out in the manner that had been finally adopted on the blackboard. A further study of the figure would bring out interesting facts, which also were noted by the pupils in their books.

To encourage accuracy, and also to make supervision of work easily possible, every construction done by the pupils was reduced sooner or later to measurement, the results obtained were given out by the pupils in answer to their names, and were noted down, and then the pupils were told what the result should have been had all their work been exact. The pupils were seated in alphabetical order, so that miraculous coincidences of results could be readily detected and investigated. With a view to the work to be done by the pupils in future sessions in the physics laboratory, they were made to enter the results of their measurements in neat tabular forms.

§ 5. Before giving any details of the propositions discussed, I wish to say that I did not attempt to draw up a systematic course beforehand; I simply made up my mind about a few groups of notions that I thought I could introduce, and the amount of time I gave to any one set of ideas was determined by the amount of useful interest I could arouse in those ideas. Whenever I found the interest likely to wane, I prepared for the introduction of a fresh discussion; the group of notions dropped being, if necessary, resumed and extended later on by way of revisal.

§ 6. I shall now give pretty full notes of two of the groups of ideas discussed (the construction and measurement of a triangle, and Euclid I. 32, with consequences) by way of specimen, and then brief notes of the rest of the work done in the twenty-five lessons gone through up to the date of the reading of this paper.

§ 7. Notes of Lessons.

Lesson 1. *Make a* Δ *with sides* 2″, $2\frac{1}{2}$″, 3″ *long* (3 *measurements.*)

(i) *having* 2″ *side as base.*

(ii) ,, $2\frac{1}{2}$″ ,,

(iii) ,, 3″ ,,

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Number of measurements required to determine any rectilineal figure investigated and tabulated as follows:—

No. of Sides	3 4 5 6.....n
No. of Meas^{ts}	3 5 7 9.....$2n-3$

Discussion on way to find area of Δ.

Area of rectangle.

Area of Δ shown to be = ½ circumscribing rectangle.

Rule deduced for finding Δ area, and noted.

Necessary for above to be able to draw a ⊥ from a point on a line.

Two methods given.

Lesson 2. *Draw ⊥rs from the vertices of the Δs in lesson* 1, *and hence find the areas of the three Δs.*

Enter results in tabular form as shown on the blackboard.

Lesson 3. *Make a quadrilateral ABCD having given* $AB = 5''$, $BC = 3''$, $CD = 3\frac{1}{2}''$, $DA = 5\frac{1}{2}''$, $AC = 6''$ (5 *measurements*).

Find its area by finding the area of each Δ.

How to find the area of any rectilineal figure.

. . . . . .

Lesson 8. Show how to make an ∠ on one part of the page equal to an ∠ on another part, then give the following exercise:

Make a ΔABC of any size and shape, and at ∠A copy down ∠s equal to B and C. Let these be BAD and DAE.

Produce CA to any point F.

Take a show of hands as to the coincidence or non-coincidence of AF with AE.

All but a very few will declare for coincidence or very near coincidence.

Note result (Euc. I., 32).

Note how to draw a line parallel to another.

Lesson 9. An equilateral Δ is equiangular.

Hence ∠ of equilateral Δ = ⅓ of 180° = 60°.

How to make ∠s of 60°, 30°, 15°, &c. (How to bisect an ∠ has already been shown.)

Make an ∠ of 60° *and one of* 30° *beside it.*

Show that this is just the method (already given without explanation) of drawing a ⊥ at end of line.

§ 8. The other chief points taken up were:

a. The construction of triangles and quadrilaterals from various data (the angles given being always 90°, 45°, 60°, 30°, and the like).

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b Illustrations of Euc. I., 47.

c Discussion of figures of the same shape (their areas, &c.).

The principles of mapping and drawing to scale.

How to measure the breadth of a river without crossing it.

d Use of squared paper to find areas. A Δ described and its area found (1) by counting the whole squares included and estimating the broken ones; (2) by the ordinary method. Results compared. Area of circle found in this way.

§ 9. To the foregoing I am permitted now (March 1888) to add a few remarks.

The course above described is neither complete nor systematic. If an idea came in the regular course, good and well; if a digression was necessary we digressed. We had no text-book to follow, and no examination to prepare for. As it happened, the pupils were examined after all, and it may encourage others to know that the work was characterised as "an excellent special course."

This session the course, still subject to modification, is being repeated to four classes similar to last year's three. A problem discussed this year with great interest was the finding of the distance of an inaccessible object by actual work in the open air. The pupils got out in batches of ten (the rest of the class working at another problem). A chain was used to measure the base line, and the base angles were taken by the eye applied to a ruler laid along the paper. Precautions were taken to have the books in the proper position. Further work of this kind is to be done.

Eighth Meeting, June 10th, 1887.

GEORGE THOM, Esq., LL.D., President, in the Chair.

Note on Milner's Lamp.

By Professor TAIT.

This curious device is figured at p. 149 of De Morgan's *Budget of Paradoxes*, where it is described as a "hollow semi-cylinder, *but not with a circular curve*," revolving on pivots. The form of the cylinder is

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such that, whatever quantity of oil it may contain, it turns itself till the oil is flush with the wick, which is placed at the edge.

Refer the "curve" to polar coördinates, r and θ; the pole being on the edge, and the initial line, of length a, being drawn to the axis. Then if θ_0 correspond to the horizontal radius vector, β to any definite radius vector, it is clear that the couple due to the weight of the corresponding portion of the oil is proportional to

$$\int_{\theta_0}^{\beta} r^2 d\theta \left(a\cos\theta_0 - \frac{2}{3} r\cos(\theta - \theta_0) \right).$$

This must be balanced by the couple due to the weight of the lamp, and of the oil beyond β; and this, in turn, may be taken as proportional to

$$\cos(\alpha + \theta_0).$$

Thus the equation is

$$a\cos\theta_0 \int_{\theta_0}^{\beta} r^2 d\theta - \frac{2}{3}\left(\cos\theta_0 \int_{\theta_0}^{\beta} r^3 \cos\theta d\theta + \sin\theta_0 \int_{\theta_0}^{\beta} r^3 \sin\theta d\theta \right) = b^3 \cos(\alpha + \theta_0).$$

Differentiating twice with respect to θ_0, and adding the result to the equation, we have (with θ now put for θ_0)

$$2ar^2\sin\theta - 2ar\frac{dr}{d\theta}\cos\theta + 2r^2\frac{dr}{d\theta} = 0.$$

Rejecting the factor r, and integrating, we have

$$r^2 = 2ar\cos\theta + C.$$

This denotes a *circular* cylinder, in direct contradiction to De Morgan's statement!

As it was clear that this result, involving only one arbitrary constant, could not be made to satisfy the given differential equation for all values of b, α, and β, I fancied that it could not be the complete integral. I therefore applied to Prof. Cayley, who favoured me with the following highly interesting paper. It commences with the question I asked, and finishes with an unexpectedly simple solution of Milner's problem.

It appears clear that De Morgan did not know the solution, for the curve he has sketched is obviously one of continued curvature—and he makes the guarded statement that a friend "vouched for Milner's Lamp."

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On a Differential Equation and the Construction of Milner's Lamp.

By Professor CAYLEY.

What sort of an equation is

$$b^3\cos(\alpha+\theta) = a\cos\theta\int_\theta^\beta r^3 d\theta - \frac{2}{3}\left\{\cos\theta\int_\theta^\beta r^3\cos\theta d\theta + \sin\theta\int_\theta^\beta r^3\sin\theta d\theta\right\}? \quad (1)$$

Write $$X = \int_\theta^\beta r^3 d\theta,\ Y = \int_\theta^\beta r^3\cos\theta d\theta,\ Z = \int_\theta^\beta r^3\sin\theta d\theta, \quad (2)$$

and start with the equations

$$d\theta = \frac{dX}{-r^3} = \frac{dY}{-r^3\cos\theta} = \frac{dZ}{-r^3\sin\theta} \quad (3)$$

$$\left(\frac{d^2}{d\theta^2}+1\right)\left\{a\cos\theta.X - \frac{2}{3}(Y\cos\theta + Z\sin\theta)\right\} = 0. \quad (4)$$

This last gives $$(r - a\cos\theta)dr + ar\sin\theta.d\theta = 0, \quad (5)$$

and the system thus is

$$d\theta = \frac{dX}{-r^3} = \frac{dY}{-r^3\cos\theta} = \frac{dZ}{-r^3\sin\theta} = \frac{(r-a\cos\theta)dr}{-ar\sin\theta}, \quad (6)$$

viz., this is a system of ordinary differential equations between the five variables θ, r, X, Y, Z: the system can therefore be integrated with 4 arbitrary constants, and these may be so determined that for the value β of θ, X, Y, Z shall be each $= 0$; and r shall have the value r_0.

But this being so, from the assumed equations (3) and (4) we have

$$X = \int_\theta^\beta r^3 d\theta,\ Y = \int_\theta^\beta r^3\cos\theta d\theta,\ Z = \int_\theta^\beta r^3\sin\theta d\theta$$

and further (by integration of 4)

$$L\cos\theta + M\sin\theta = a\cos\theta.X - \frac{2}{3}(Y\cos\theta + Z\sin\theta).$$

Where L and M denote properly determined constants: viz., the conclusion is that r, X, Y, Z admit of being determined as functions of θ and of an arbitrary constant r_0, in such wise that

$$a\cos\theta.X - \frac{2}{3}(Y\cos\theta + Z\sin\theta)$$

shall be a function of θ, of the proper form $L\cos\theta + M\sin\theta$, but not so that it shall be the precise function $b^3\cos(\alpha+\theta)$. To make it have

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this value we must have $L = b^3\cos\alpha$, $M = -b^3\sin\alpha$ (where L, M are given functions of a, β, r_0), *i.e.*, we must have *two* given relations between a, b, α, β, r_0: or treating r_0 as a disposable constant we must have *one* given relation between a, b, α, β.

The equation $d\theta = \frac{r - a\cos\theta}{-a r\sin\theta} dr$ gives $r^2 - 2ar\cos\theta = C$. ($C = r_0^2 - 2ar_0\cos\beta$). There would be considerable difficulty in working the question out with r_0 arbitrary, but we may do it easily enough for the particular value $r_0 = 0$ or $r_0 = 2a\cos\beta$, giving $C = 0$ and $\therefore\ r = 2a\cos\theta$: and we ought in this case to be able to satisfy the given equation not in general but with *two* determinate relations between the constants a, b, α, β.

We have

$$\int \cos^3\theta d\theta = \frac{1}{2}\theta + \frac{1}{4}\sin 2\theta$$

$$\int \cos^4\theta d\theta = \frac{3}{8}\theta + \frac{1}{4}\sin 2\theta + \frac{1}{32}\sin 4\theta$$

$$\int \cos^3\theta\sin\theta d\theta = -\frac{1}{4}\cos^4\theta$$

And thence

$$a\cos\theta.X - \frac{2}{3}(Y\cos\theta + Z\sin\theta)$$

$$= 4a^3\cos\theta\left\{\frac{1}{2}(\beta - \theta) + \frac{1}{4}(\sin 2\beta - \sin 2\theta)\right\}$$

$$-\frac{16}{3}a^3\cos\theta\left\{\frac{3}{8}(\beta - \theta) + \frac{1}{4}(\sin 2\beta - \sin 2\theta) + \frac{1}{32}(\sin 4\beta - \sin 4\theta)\right\}$$

$$-\frac{16}{3}a^3(\sin\theta\left\{ \qquad\qquad\qquad\qquad -\frac{1}{4}(\cos^4\beta - \cos^4\theta)\right\}$$

$$= -\frac{1}{3}a^3\cos\theta(\sin 2\beta - \sin 2\theta)$$

$$-\frac{1}{6}a^3\cos\theta(\sin 4\beta - \sin 4\theta)$$

$$+\frac{4}{3}a^3\sin\theta(\cos^4\beta - \cos^4\theta)$$

Where the terms containing β are readily reduced to $\frac{4}{3}a^3\cos^3\beta\sin(\theta - \beta)$; hence also the terms without β disappear of themselves: and we have

$$a\cos\theta.X - \frac{2}{3}(Y\cos\theta + Z\sin\theta) = \frac{4}{3}a^3\cos^3\beta.\sin(\theta - \beta),$$

which may be put $= b^3\cos(\theta + \alpha)$:

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viz., this will be so if we have the *two* relations

$$\alpha = \frac{\pi}{2} - \beta \text{ ; and } b^3 = -\frac{4}{3}a^3\cos^3\beta.$$

I make (see fig. 84) Milner's lamp, with a circular section, β arbitrary, but a segment AM ($\angle$ SAM $= \beta$) made solid. G in the line SG at right angles to AM is the C.G. of the lamp, and G′ the C.G. of the oil.

And this seems to be the *only* form—for the pole of r must, it seems to me, be *on* the bounding circle—viz., in the equation $r^2 - 2ar\cos\theta = C$, we must have $C = 0$.

An Exercise on Logarithmic Tables.

By Professor Tait.

In reducing some experiments, I noticed that the logarithm of 237 is about 2.37 ... . Hence it occurred to me to find in what cases the figures of a number and of its common logarithm are identical :—*i.e.*, to solve the equation

$$\log_{10} x = x/10^m,$$

where m is any positive integer.

It is easy to see that, in all cases, there are two solutions ; one greater than, the other less than, ϵ. This follows at once from the position of the maximum ordinate of the curve

$$y = (\log x)/x.$$

The smaller root is, for

$$m = 1,\ x = 1{\cdot}371288 \quad \ldots \quad \ldots$$
$$m = 2,\ x = 1{\cdot}023855 \quad \ldots \quad \ldots$$

For higher values of m, it differs but little from 1, and the excess may be calculated approximately from

$$y - y^2/2 + \quad \ldots \quad = (1+y)\log_\epsilon 10/10^m.$$

Ultimately, therefore, the value of the smaller root is

$$1.00 \quad \ldots \quad \ldots \quad 0230258 \quad \ldots \quad \ldots$$

where the number of cyphers following the decimal point is $m - 1$.

The greater root must have $m + p$ places of figures before the decimal point ; p being unit till $m = 9$, thenceforth 2 till $m = 98$, 3 till $m = 997$, &c. Thus, for example, if $m > 8 < 98$ we may assume

$$x = (m+1)10^m + y$$

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so that

$$\log_{10}\frac{m+1}{10}+\log_{10}\left(1+\frac{y}{(m+1)10^m}\right)=\frac{y}{10^m}$$

which is easily solved by successive approximations.

But it is simpler, and forms a capital exercise, to find, say to six places, the greater root, by mere inspection of a good Table of Logarithms.

Thus we find, for instance,

m	x
17	$182{,}615.10^{13}$
18	$192{,}852.10^{14}$
96	$979{,}911.10^{92}$
97	$989{,}956.10^{93}$

Geometrical Proof of the Tangency of the Inscribed and Nine-Point Circles.

By William Harvey, B.A.

S (fig. 85) is the circumscribed centre, and O the orthocentre of the triangle ABC; AX the perpendicular from A on BC, and P the middle point of BC.

SP produced bisects the arc BC in V, and I, the centre of the inscribed circle, lies on AV, and is so situated that $AI.IV = 2Rr$. (*See Note*). Also the angle XAV = angle AVS = angle SAV.

N, the centre of the nine-point circle, bisects the distance OS, and the circumference passes through P, X and L, the middle point of AO. Hence N bisects both LP and OS, and

$$SP = OL = AL;$$

therefore LP is parallel to AS.

NHM is a radius of the nine-point circle, bisecting the chord XP in H, and the arc XP in M; ID is a radius of the inscribed circle.

Since the chord XMP is bisected at M,

$$\text{the angle XPM} = \frac{1}{2}\text{angle XLP},$$

$$= \frac{1}{2}\text{ angle XAS},$$

$$= \text{angle XAV or AVS}.$$

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Hence, if through I we draw a straight line (not shown in the figure) parallel to BC to meet AX and SV, the segments of this line are respectively equal to XD and PD, and we have by similar triangles

$$XD : IA = HM : MP;$$
$$PD : IV = HM : MP;$$

and therefore

$$XD.PD : IA.IV = HM^2 : MP^2.$$

But $IA.IV = 2Rr$, and $MP^2 = R.HM$.

Hence $$XD.PD = 2r.HM.$$

But $$HP^2 - HD^2 = XD.PD = 2r.HM.$$

Now
$$\begin{aligned} IN^2 &= (NH - r)^2 + HD^2, \\ &= (NH - r)^2 + HP^2 - 2r.HM, \\ &= NH^2 + HP^2 - 2r.(NH + HM) + r^2, \\ &= \frac{R^2}{4} - rR + r^2; \end{aligned}$$

Or
$$IN = \frac{R}{2} - r.$$

Whence the tangency of the circles is evident.

Note.—The following is a simple proof of the theorem $AI.IV = 2Rr$, assumed in the foregoing proof :—

Draw IE perpendicular to AC, and ST perpendicular to VC.

Since angle AVC = angle B = 2 angle IBC, I lies on the circumference of a circle whose centre is V and radius VB or VC.

Hence $$IV = VC.$$

Again, the triangles AIE, SVT are clearly similar;

$$\therefore AI : IE = SV : VT.$$

But $$VT = \frac{1}{2}VC = \frac{1}{2}IV;$$

$$\therefore AI.IV = 2Rr.$$

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Fig. 1

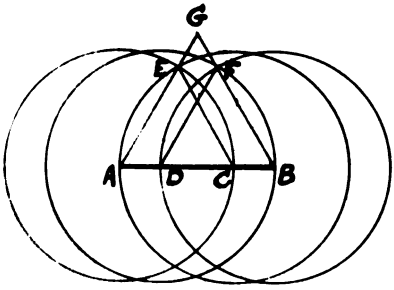


Fig. 2

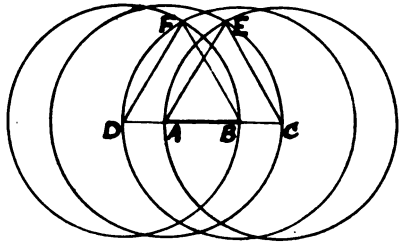


Fig. 3

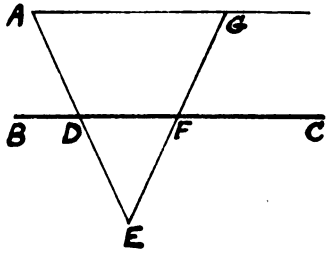


Fig. 4

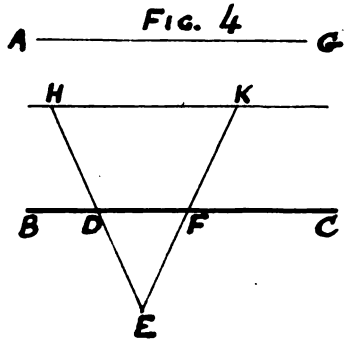


Fig. 5

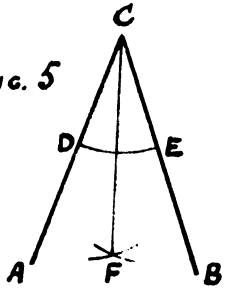
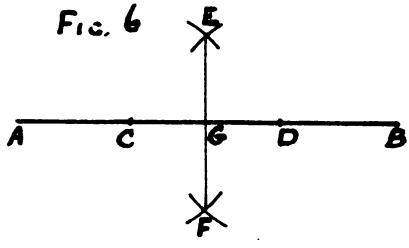


Fig. 6



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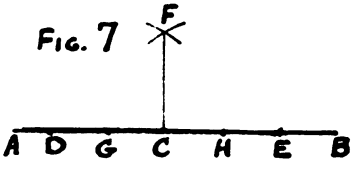

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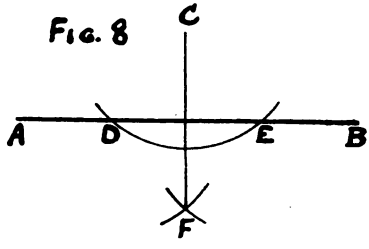

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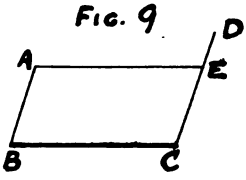

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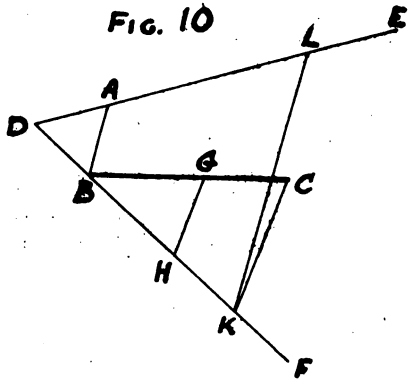

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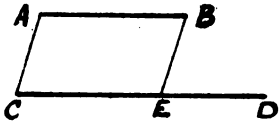

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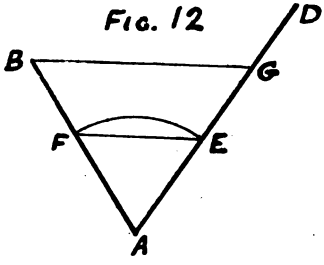

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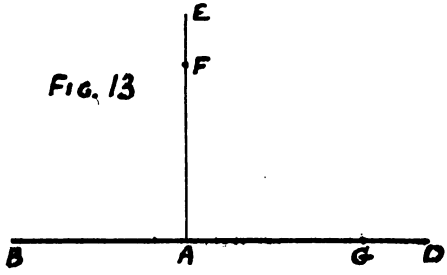

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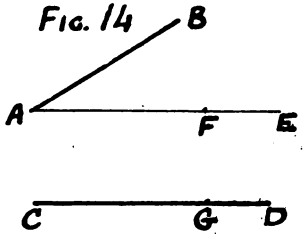

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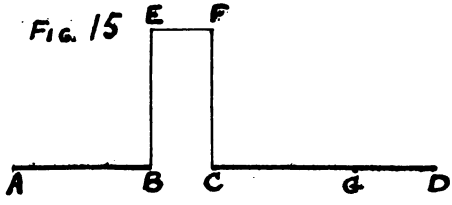

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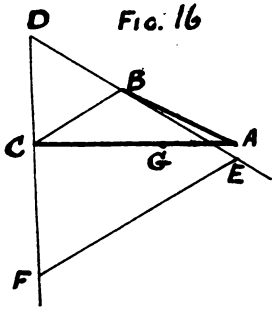

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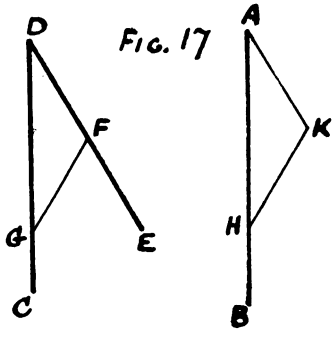

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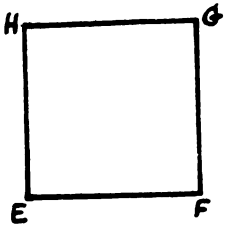

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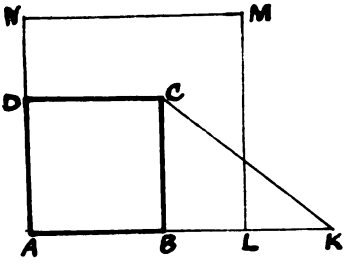


Fig. 19

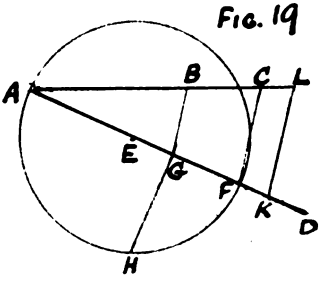
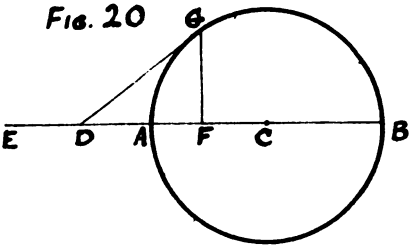


Fig. 20



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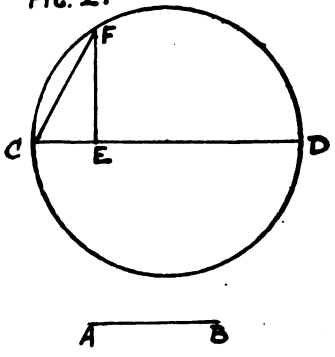

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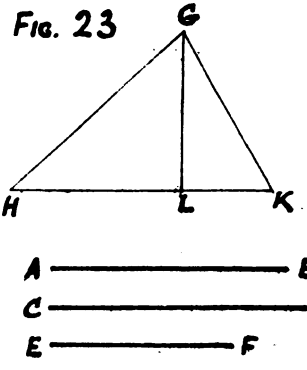

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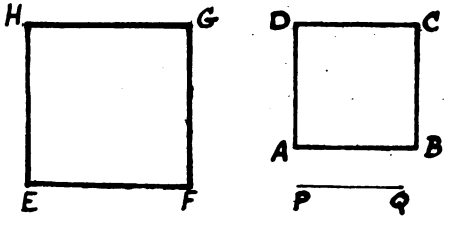

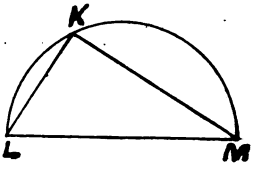

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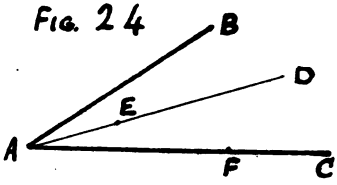

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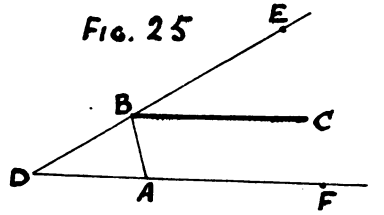

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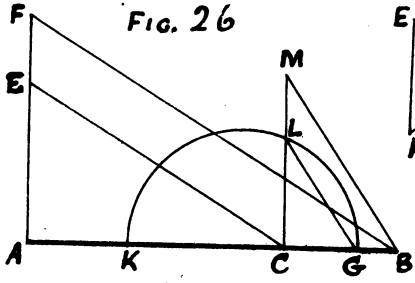

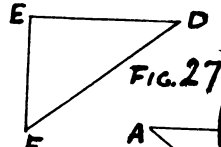

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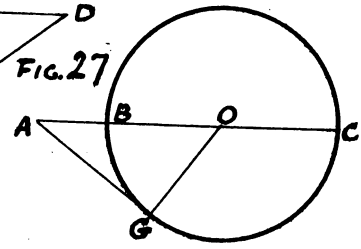

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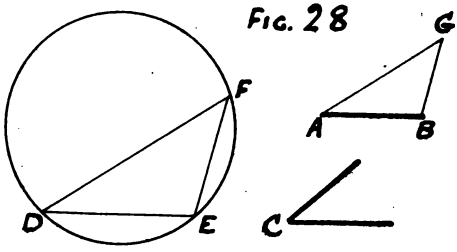

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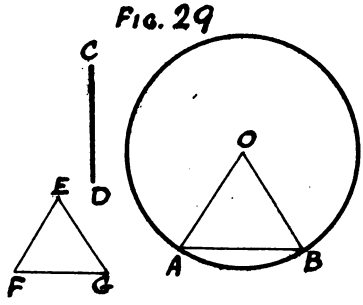

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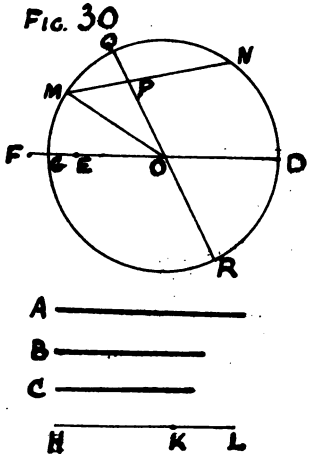

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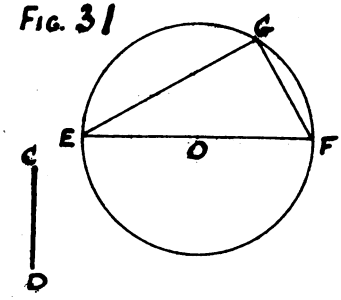

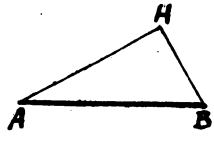

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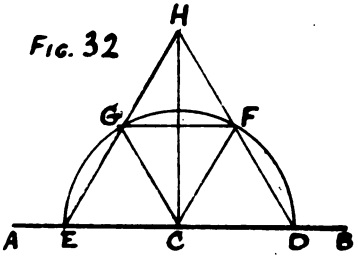
Fig. 32



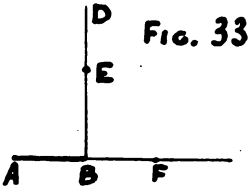
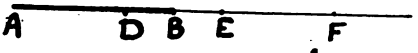


Fig. 33

Fig. 34



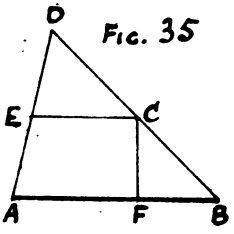
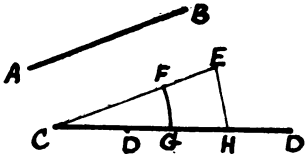


Fig. 35

Fig. 36



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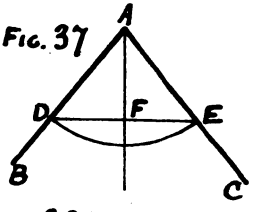


Fig. 38

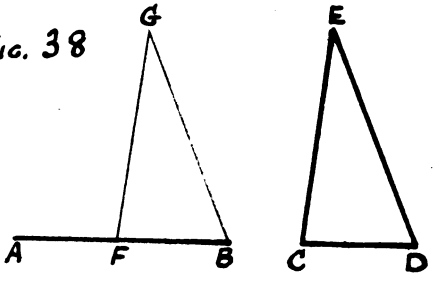
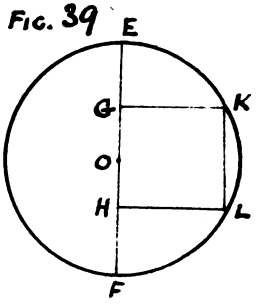
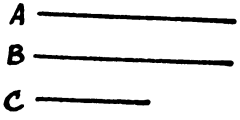


Fig. 39





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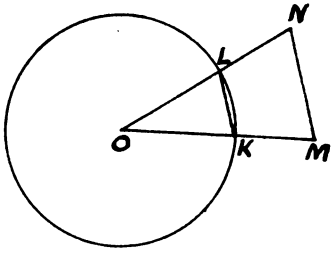

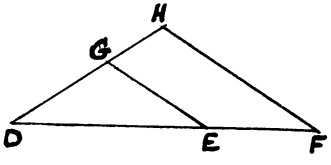

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FIG. 41

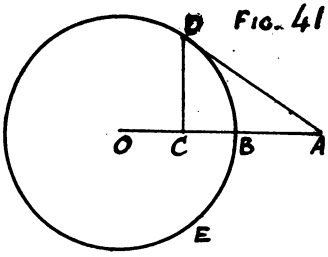


FIG. 42

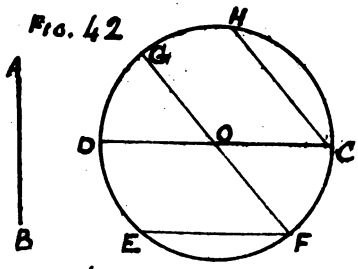
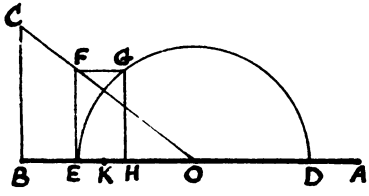


FIG. 43



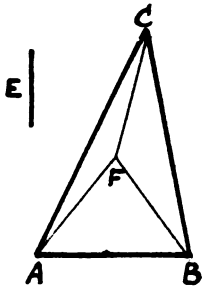


FIG. 44

FIG. 45

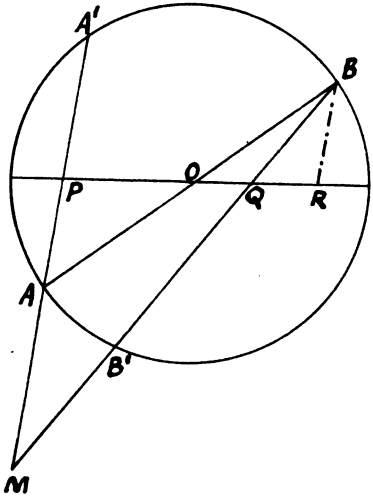


FIG. 46

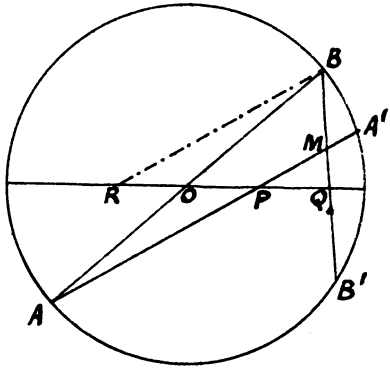


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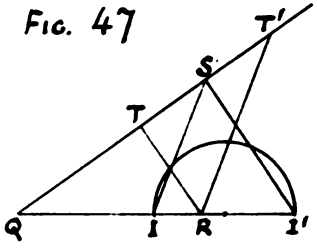
Fig. 47



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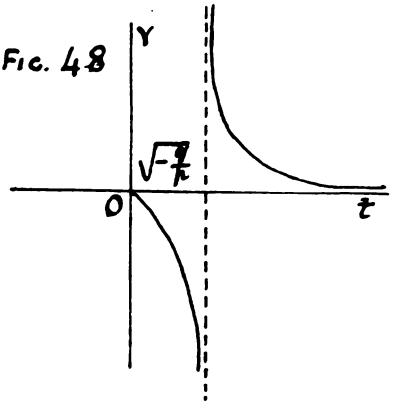

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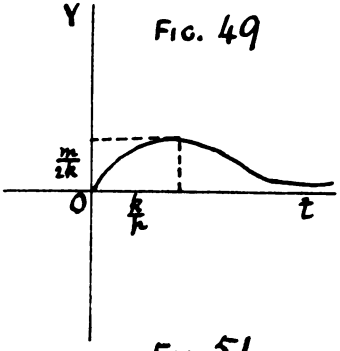

Fig. 50

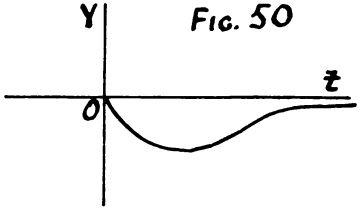

Fig. 51

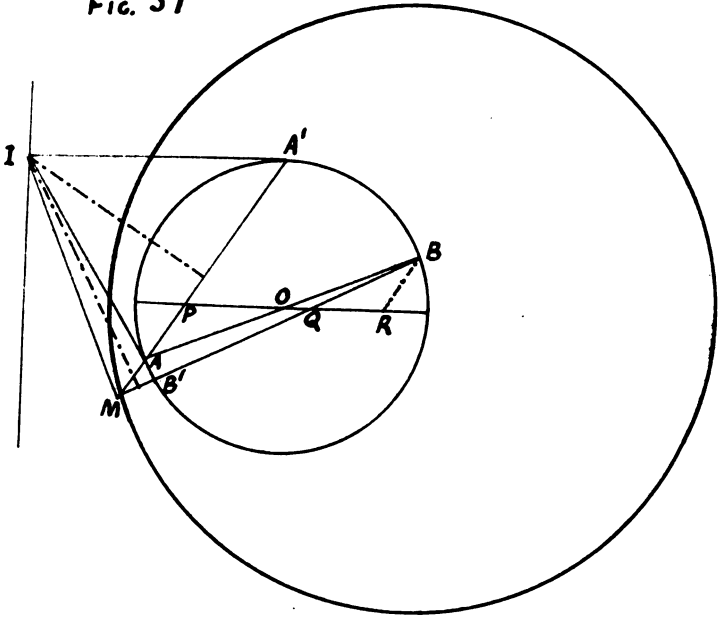

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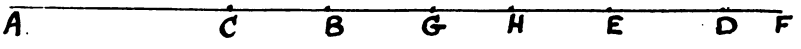
FIG. 52



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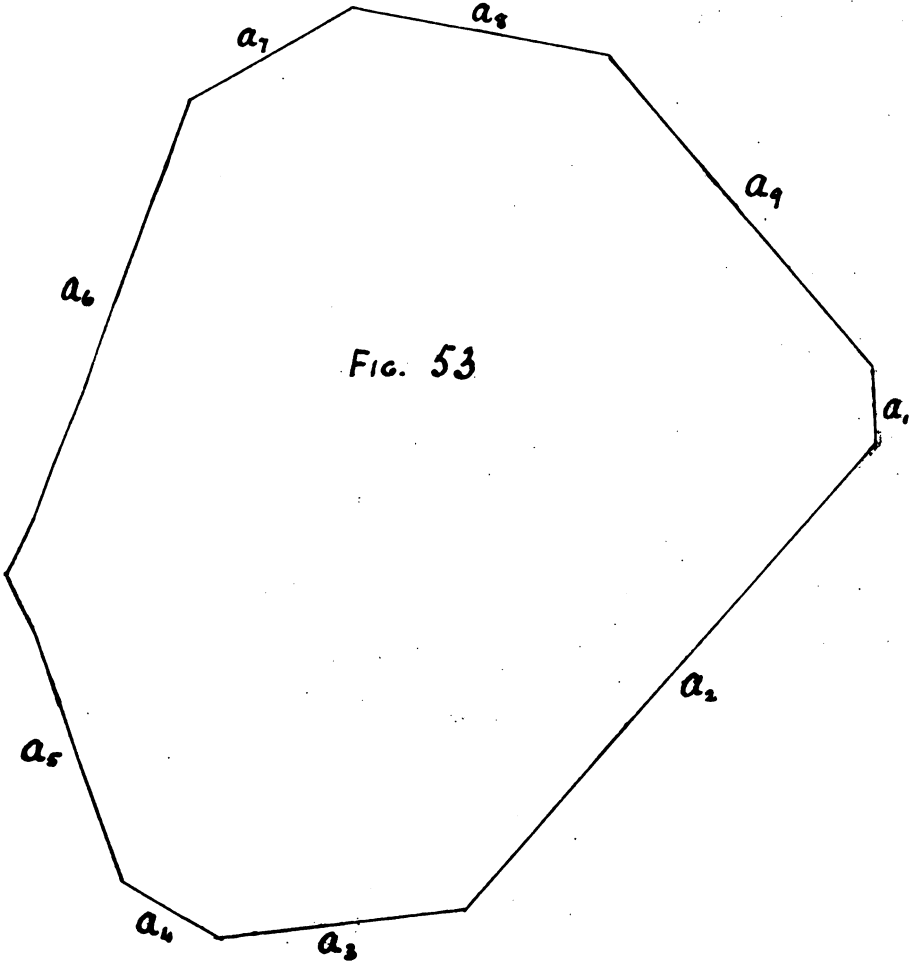

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FIG. 54

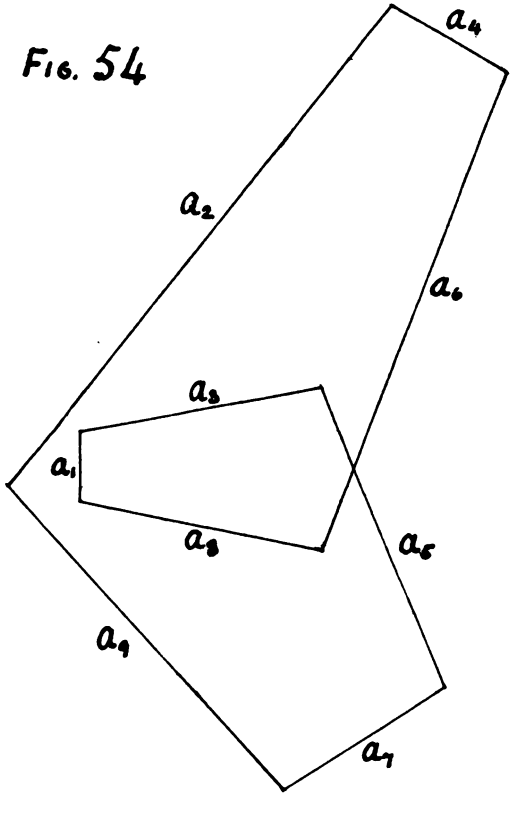

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FIG. 55

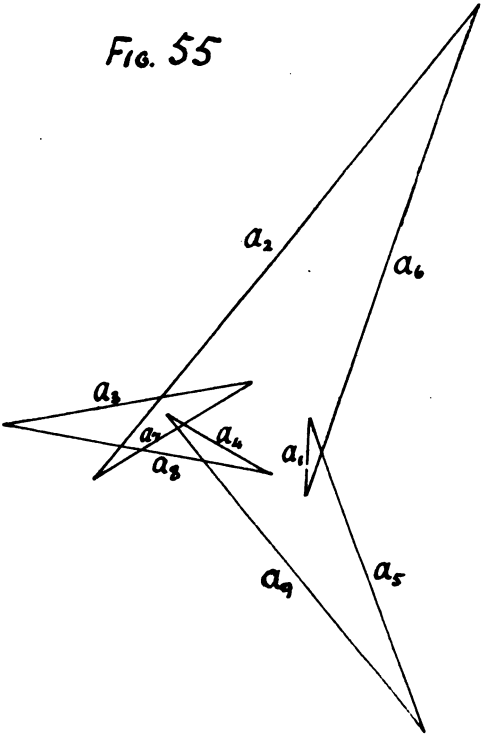

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Fig 56

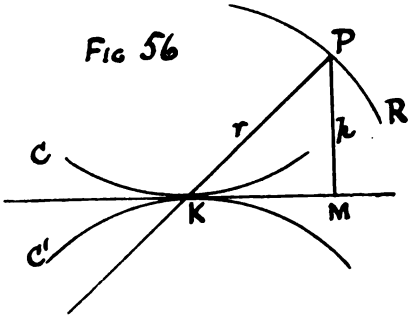

Fig 57

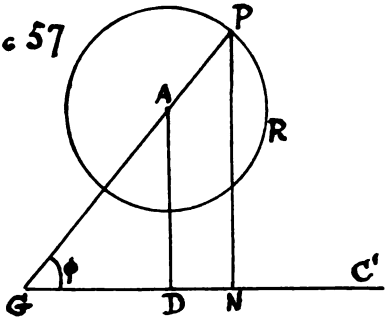

Fig 58

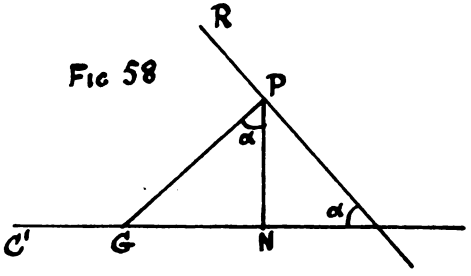

Fig 59

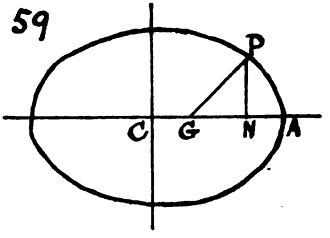

Fig 60

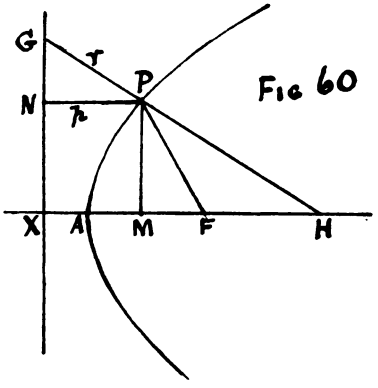

Fig 61

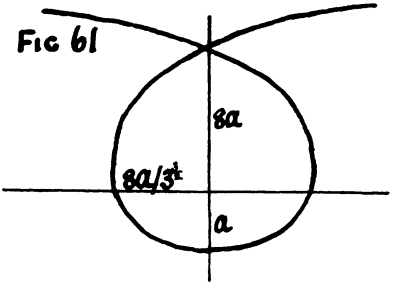

Fig 62

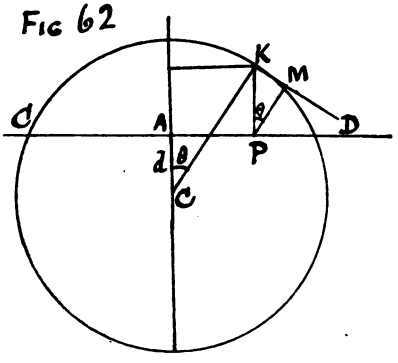

Fig 63

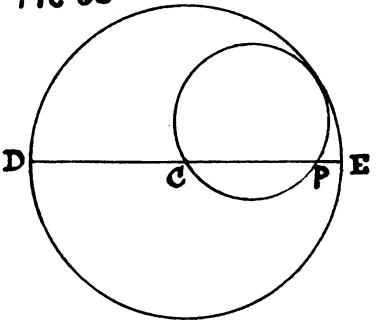

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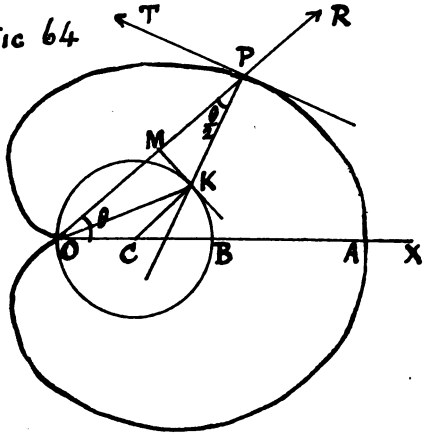
Fig 64



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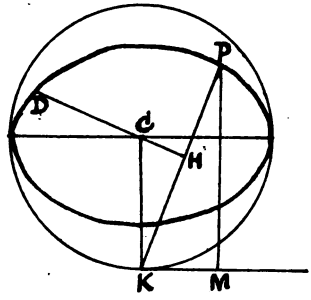

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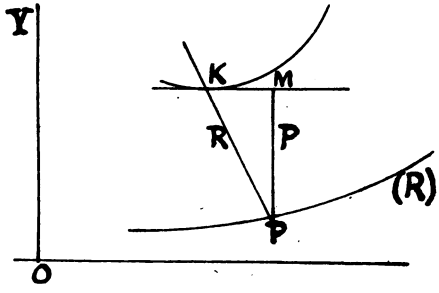

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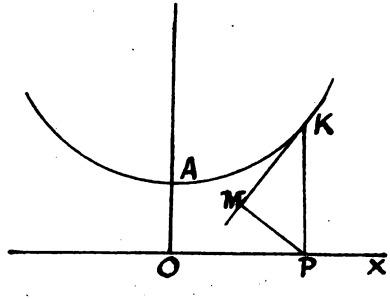

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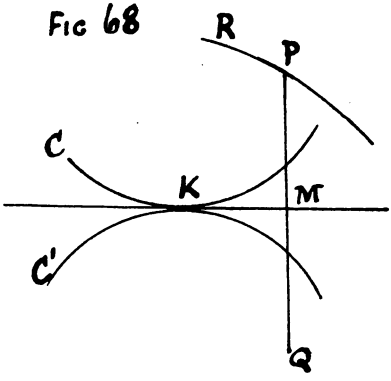

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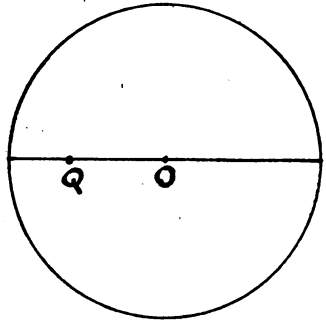

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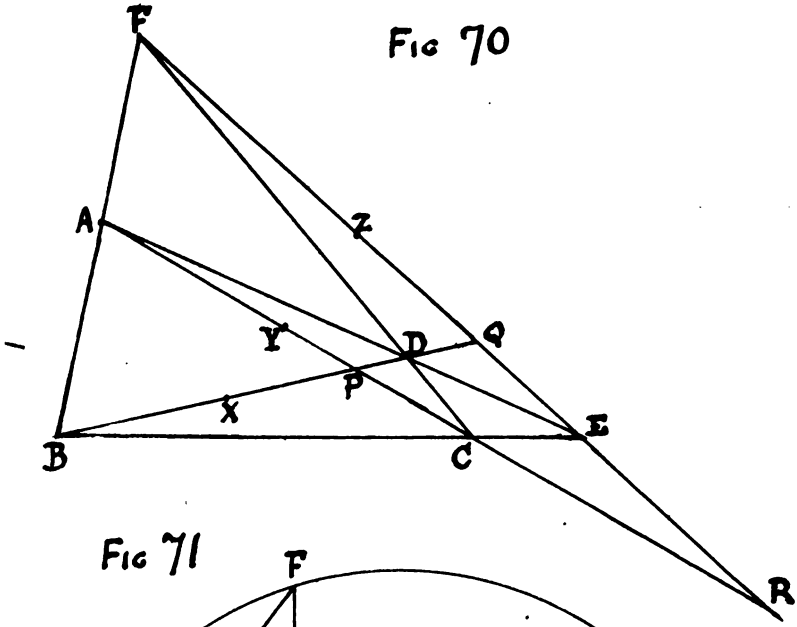
Fig 70



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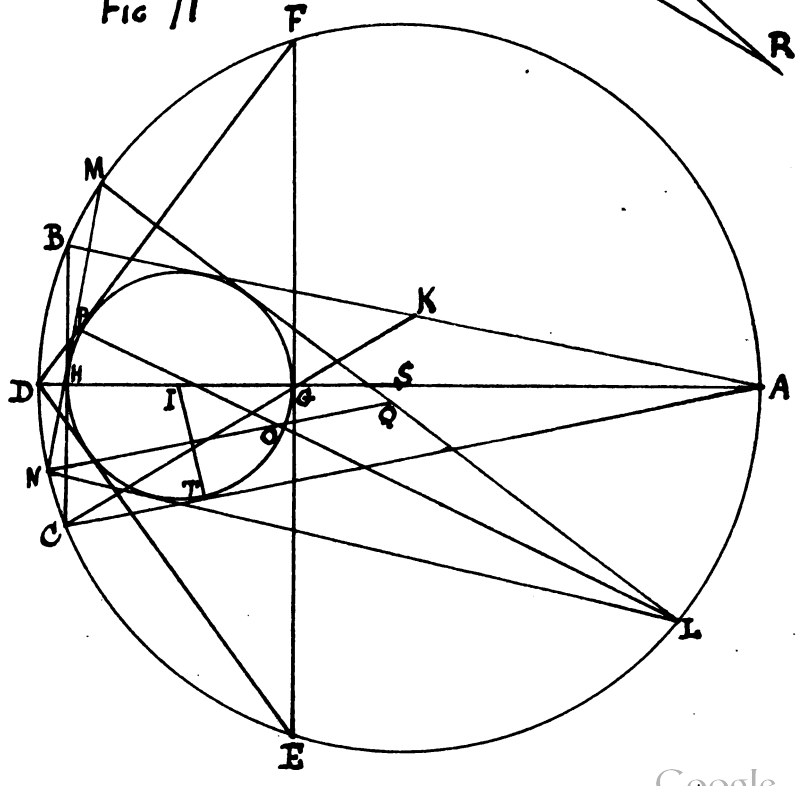

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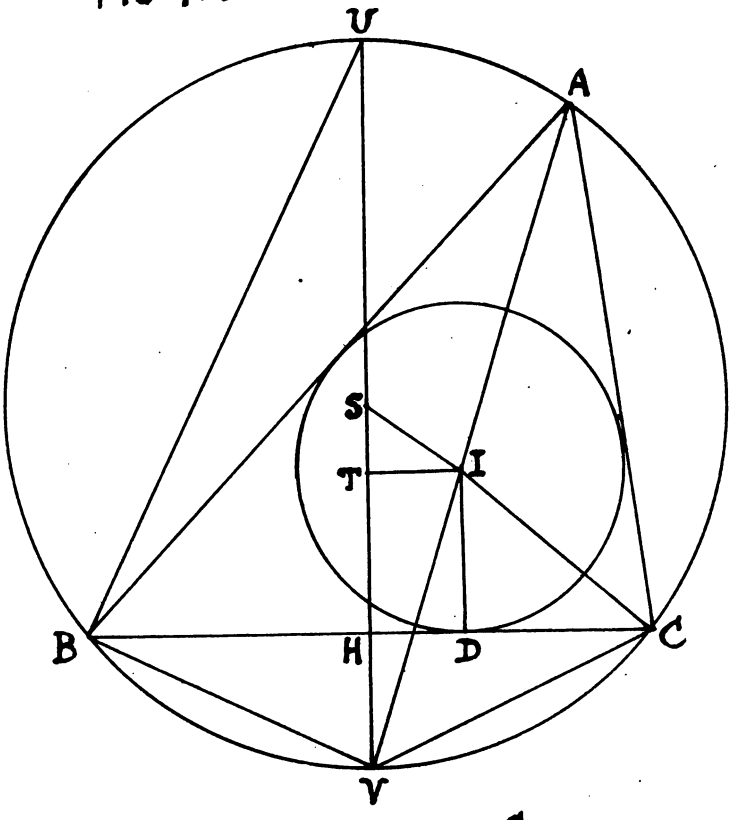

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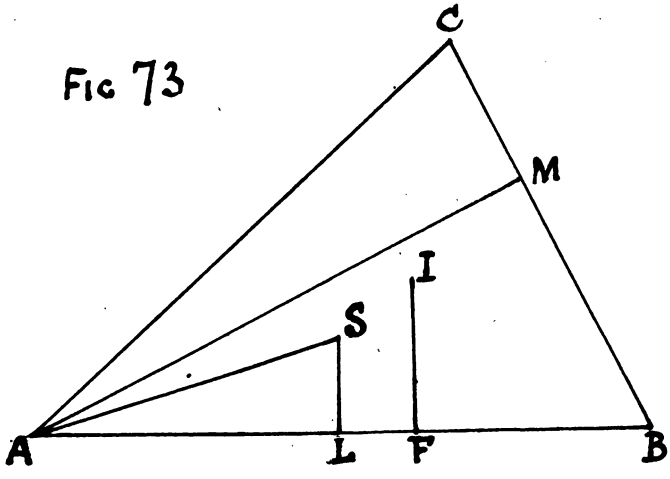

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Fig 74

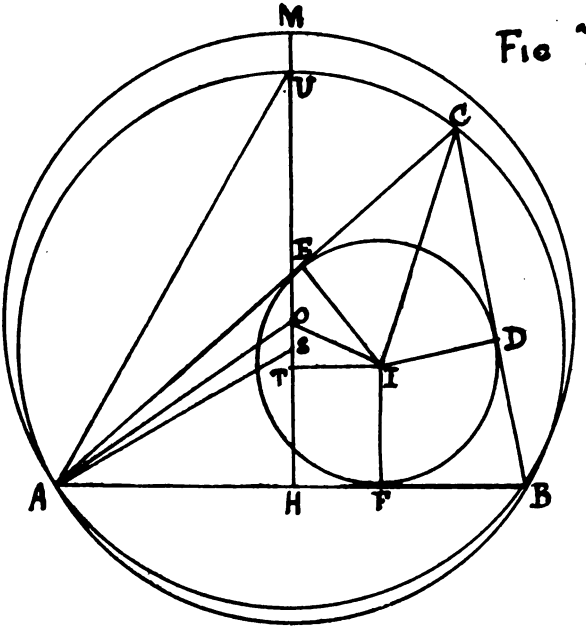
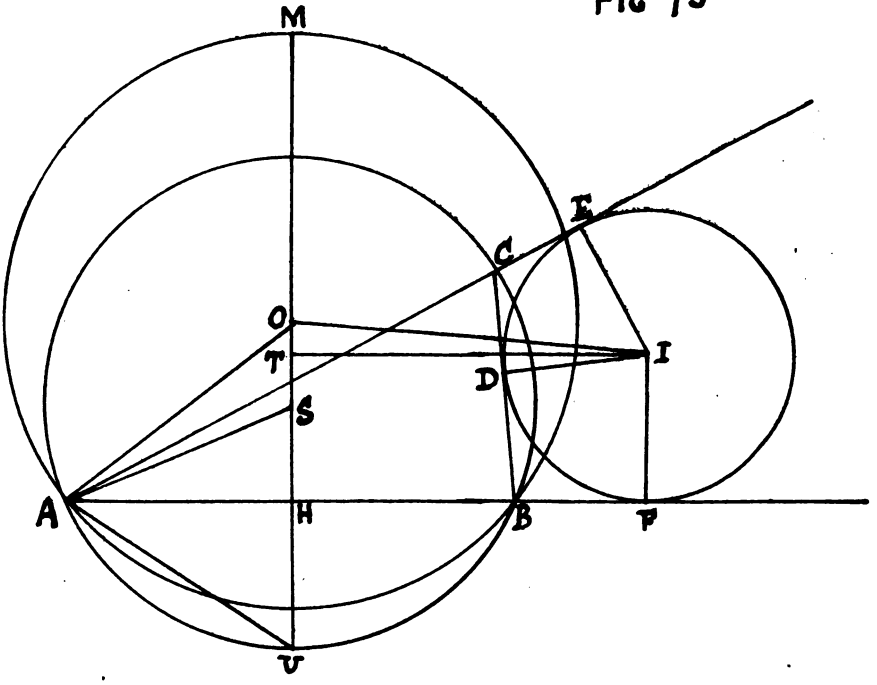


Fig 75



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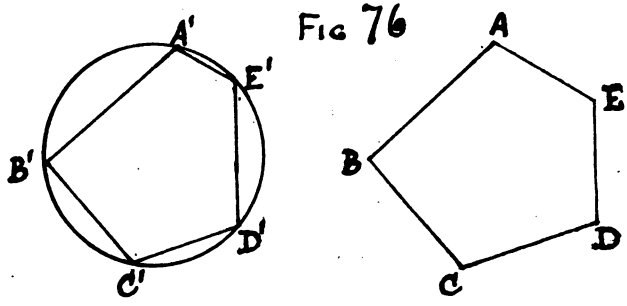

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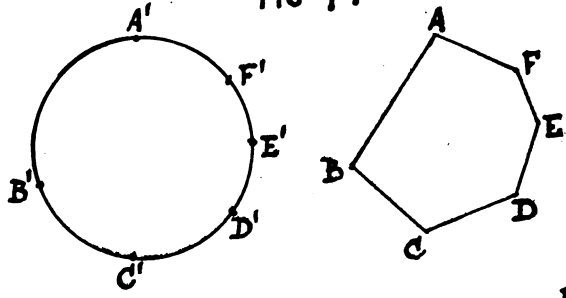

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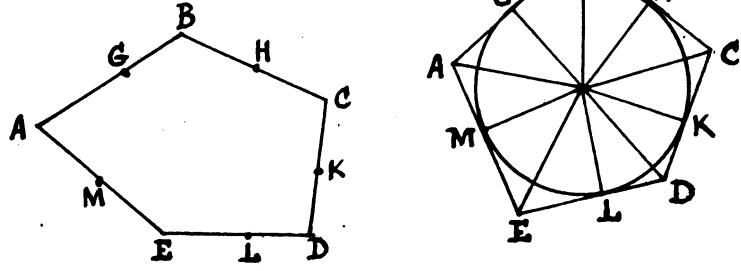

Fig 79

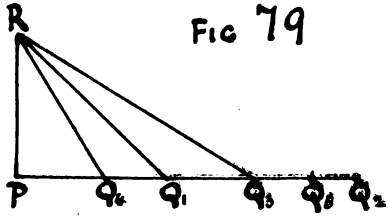

Fig 80

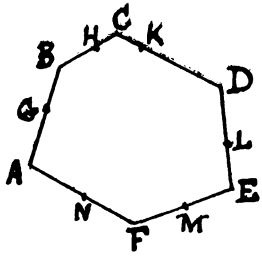

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FIG 81

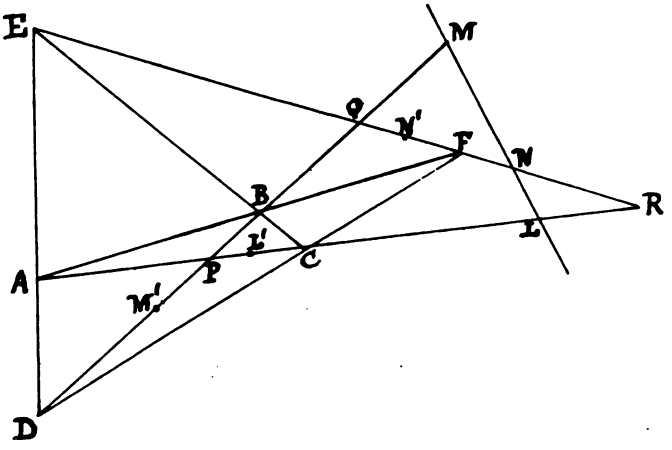

FIG 82

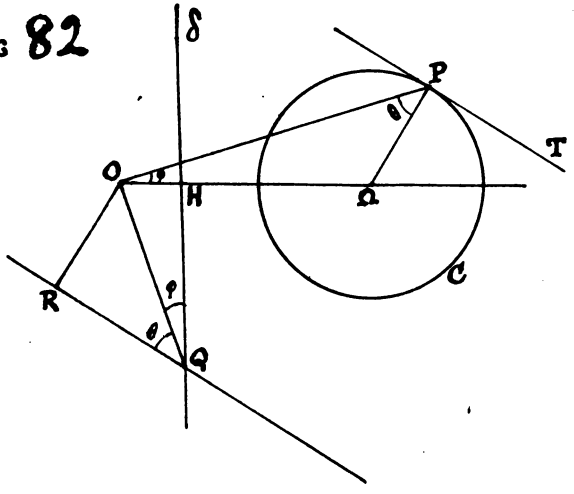

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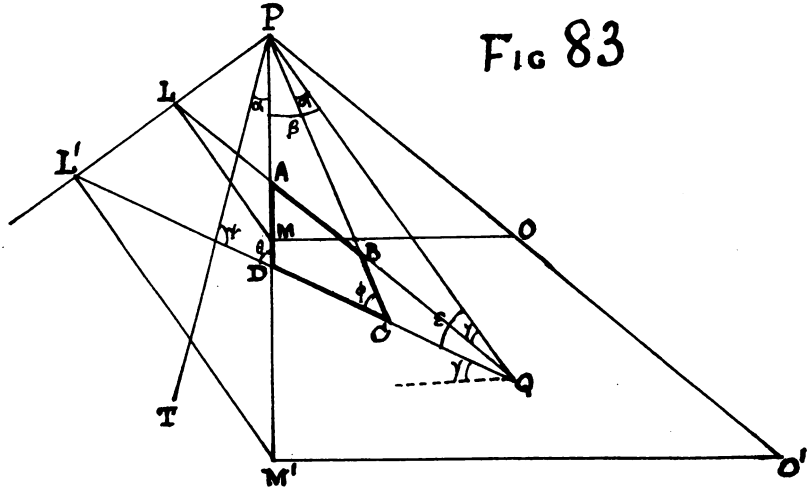
Fig 83



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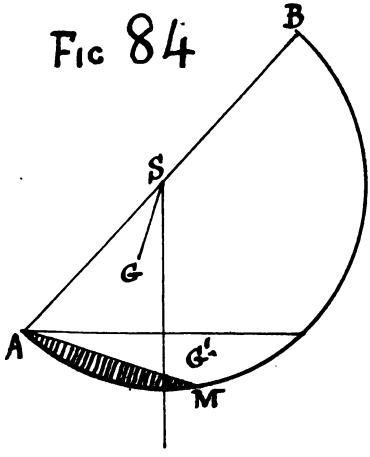

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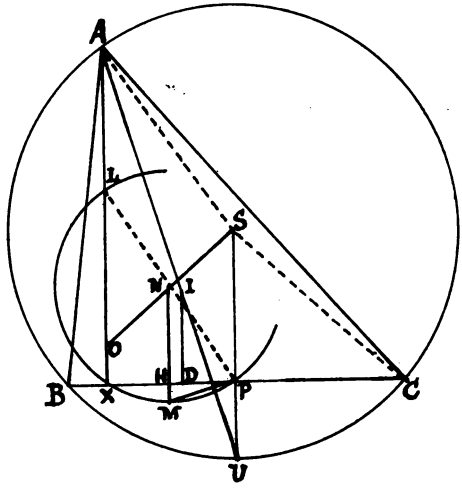

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